Homework 1.1 Solution

DE: $u_{tt} + c\, u_t = u_{xx},\ c > 0$
BC: $u(0,t) = u(\pi,t) = 0,\ u_t(x,0) = 0$
IC: $u(x,0) = u_0(x)$

Try $u(x,t) = X(x)T(t)$, without loss of generality we can require $T(0) = 1$. With these definitions, our system becomes:

DE: $T''/T + c\, T'/T = X''/X$
BC: $X(0) = X(\pi) = 0,\ T(0) = 1,\ T'(0) = 0$
IC: $X(x) = u_0(x)$

The LHS and RHS of the DE are independent which implies both sides must be equal to a constant, say λ. There are 2 cases to consider:

1. $\lambda = 0$
   $X'' = 0 \Rightarrow X = a + bx$
   BC $\Rightarrow X \equiv 0$

2. $\lambda = -\alpha^2$ (with α possibly complex)
   $X'' = -\alpha^2 X \Rightarrow X = a\cos\alpha x + b\sin\alpha x$
   BC $\Rightarrow X(0) = a = 0,\ X = b\sin\alpha x$
   BC $\Rightarrow X(\pi) = b\sin\alpha\pi = 0 \Rightarrow \alpha\pi = n\pi \Rightarrow \alpha = n$ (real)

Only the last case has a non-trivial solution. Therefore, we have a sequence of solutions,

$$X_n = b_n \sin n\pi,\ \lambda_n = -n^2,\ n = 1, 2, 3, \ldots \tag{1}$$

Having solved for X and λ, we need to solve for T: $T'' + cT' + n^2 T = 0$. This is a constant coefficient differential equation whose solution is determined by the roots of its characteristic polynomial. To simplify the algebra, define $\gamma = \frac{c}{2}$ and $\Delta^2 = n^2 - \gamma^2$, so the roots are $r_{1,2} = -\gamma \pm \Delta i$ with Δ possibly complex. There are 2 cases to consider:

1. $\Delta = 0 \Rightarrow r_1 = r_2$
   $T = a\, e^{rt} + b\, t\, e^{rt} = e^{-\gamma t}(a' + b't)$
   BC $\Rightarrow T = e^{-\gamma t}(1 + \gamma t)$

2. $\Delta \neq 0 \Rightarrow r_1 \neq r_2$
   $T = a\, e^{r_1 t} + b\, e^{r_2 t} = e^{-\gamma t}(a'\cos\Delta t + b'\sin\Delta t)$
   BC $\Rightarrow T = e^{-\gamma t}(\cos\Delta t + \frac{\gamma}{\Delta}\sin\Delta t)$

(check that $T'(0) = 0$) If Δ is imaginary then T involves the hyper-trig functions but remains real. In all cases there is a damping factor of $e^{-\gamma t}$. Having solved for X_n and T_n we are left with the initial condition $X(x) = u_0(x)$. Looking back at X_n, if we could write $u_0(x) = \sum b_n \sin nx$ then a solution to the entire problem would be,

$$u(x,t) = \sum_{n=1}^{\infty} b_n X_n(x) T_n(t) \tag{2}$$

we recognize b_n as the fourier sin expansion of $u_0(x)$, so this is always possible and

$$b_n = \frac{2}{\pi} \int_0^{\pi} u_0(x) \sin nx \; dx \tag{3}$$

This completes the solution for all values of c and all initial conditions. If $|\gamma| < 1 \Leftrightarrow |c| < 2$ then all solutions of T will be of the same form:

$$u(x,t) = e^{-\gamma t} \sum_{n=1}^{\infty} b_n \sin(n\pi)(\cos \Delta_n t + \frac{\gamma}{\Delta_n} \sin \Delta_n t) \tag{4}$$

where $\gamma = \frac{c}{2}$, $\Delta_n^2 = n^2 - \gamma^2$, and b_n as above. The solutions for the special cases are:

- $c = 1, \; u_0 = \sin x$
  $\gamma = \frac{1}{2}, \; \alpha_1^2 = 1 - \gamma^2 = 3/4, \; u(x,t) = e^{-t/2} \sin(x)(\cos \frac{\sqrt{3}}{2} t + \frac{1}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} t)$
- $c = 1, \; u_0 = \sin 3x$
  $\gamma = \frac{1}{2}, \; \alpha_3^2 = 3^2 - \gamma^2 = 35/4, \; u(x,t) = e^{-t/2} \sin(3x)(\cos \frac{\sqrt{35}}{2} t + \frac{1}{\sqrt{35}} \sin \frac{\sqrt{35}}{2} t)$

Extra Credit.

- c is the "damping" coefficient.
- if $c = 0$ there is no damping and T shows oscillatory behavior.
- if $c < 0$ then the system explodes as $e^{-ct/2} \to \infty$.
- as c increases, the $e^{-ct/2}$ term dominates.
- if $c = 2n$ then mode n experiences "critical-damping" (no oscillations) while nodes less than n experience "over-damping".