Homework 1

1. **Analytic investigation of the wave equation in a resistant medium.**

   Wave phenomena in a resistant elastic medium are described with

$$\begin{aligned}&\ddot{u}(x,t) + c\dot{u}(x,t) - u''(x,t) = 0,\\ &u(0,t) = u(\pi,t) = 0,\\ &u(x,0) = u_0(x),\\ &\dot{u}(x,0) = 0,\\ &0 < x < \pi,\ t > 0\end{aligned}$$

   It is easily seen that the above equation is equivalent to the mass-spring-dashpot system in the discrete setting. Solve the differential equation by separation of variables.

   Consider the problem with $c = 1$ for initial conditions $u(x,0) = sin(x)$ and $u(x,0) = sin(3x)$. Use the visualization capabilities of MATLAB to present your results. Characterize the solution behavior as time passes.

   Extra credit: Investigate the influence of the constant c on the solution behavior. Comment on the solution behavior. You can motivate your comments with the mass-spring-dashpot system.

2. **Numerical investigation of interpolation errors.**

   The function

$$u = x^{1-v}, \quad 0 \le x \le 1$$

   is interpolated with the piecewise constant function $\pi_0 f$ and the piecewise linear function $\pi_1 f$. The domain $\Omega = [0,1]$ is discretized with $N = 8, 16, 32, 64$ elements Ω_i, $i = 1, \ldots, N$. Compute the interpolation errors in the L_2-norms

$$\|f - \pi_0 f\|^2_{L_2(\Omega)} = \sum_{i=1}^{N} \|f - \pi_0 f\|^2_{L_2(\Omega_i)}$$

$$\|f - \pi_1 f\|^2_{L_2(\Omega)} = \sum_{i=1}^{N} \|f - \pi_1 f\|^2_{L_2(\Omega_i)}$$

   for $v = 1/4, 1/2$ and $3/4$. Use the two-point Gauss quadrature rule for the numerical evaluation of the integrals (see the handout on interpolation). Present your results in a double logarithmic diagram.

   Use the following function for node spacing,

$$x_i = \left(\frac{i}{N}\right)^{\beta}$$

with $\beta = 1$ for each value of v. For $v = 3/4$, compute the error when $\beta = 2$ and 4. Compare these results with the error in the equidistant mesh.

Extra credit: Compute analytically the optimal value of β.