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ISOMETRIC PIECEWISE LINEAR IMMERSIONS OF TWO-DIMENSIONAL MANIFOLDS WITH POLYHEDRAL METRICS INTO $\mathbb{R}^3$

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ABSTRACT. Let M be a connected compact two-dimensional manifold (possibly with boundary) equipped with a polyhedral metric ρ, i.e., a metric such that every point has a neighborhood isometric to a neighborhood of the vertex of a cone in $\mathbb{R}^3$ with finite total angle around the vertex. Such a manifold has finitely many vertices (with total angle around each vertex different from 2π) and admits a unique (up to a homeomorphism) differential structure; this structure may be assumed to possess the property that off the vertices the metrics is determined by smooth linear elements.

Let $f_0\colon M \to \mathbb{R}^3$ *be a* C^2*-smooth immersion (or embedding). If* f_0 *is short (contracting) with respect to* ρ*, then it can be* C^0*-approximated by some isometric piecewise linear immersions* $f_i\colon (M,\rho) \to \mathbb{R}^3$*. Moreover, if* f_0 *is an embedding, then the* f_i *can be chosen to be embeddings.*

Thus, if such a manifold (M,ρ) is orientable (or nonorientable but with nonempty boundary), then it admits an isometric piecewise linear embedding in $\mathbb{R}^3$. If (M,ρ) is nonorientable and closed, it admits a piecewise linear immersion into $\mathbb{R}^3$.

By using the same constructions it is also proved that there exists a convex polyhedron in $\mathbb{R}^3$ whose surface (with solid faces and fixed combinatorial scheme) admits an isometric embedding in $\mathbb{R}^3$ as the boundary of another (nonconvex) polyhedron of larger volume.

§1. MAIN RESULTS

1.1. By a two-dimensional manifold with polyhedral metric (in brief, a *polyhedron*) we mean a metric space endowed with the structure of a connected compact two-dimensional manifold (possibly with boundary) every point x of which has a neighborhood isometric to a neighborhood of the vertex of a cone. We assume that this cone has rectifiable directrix (which is a closed curve if x is an interior point, and a simple arc if x is a boundary point).

The metric of a polyhedron is locally flat everywhere except for a finite collection of points; these points are the "true" vertices. A polyhedron with nonempty boundary contains finitely many "true" corners of the boundary (if any). The triangulations of polyhedrons that we consider consist of triangles isometric to straight triangles in $\mathbb{R}^2$. Such triangulations always exist; they are also called *developments*. All "true" vertices and corners are necessarily among the vertices of any development (in general, the latter vertices are more numerous).

A polyhedron can be defined using any of its developments.

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1.2. We recall that a map $f\colon X \to Y$ of metric spaces is called *C-short* if

$$d_Y(fx, fy) \leq C d_X(x, y)$$

for all $x, y \in X$; here d_X and d_Y denote the distance functions. A map f is called *short* (or *contracting*) if it is C-short for some $C < 1$.

1.3. Our aim here is to prove the following discrete analog of the well-known results of J. Nash [1] and N. Kuiper [2] on C^1-smooth isometric immersions; we can do this in dimension 2.

1.4. Theorem. *Every short C^2-immersion f_0 of a polyhedron M into $\mathbb{R}^3$ admits a C^0-approximation by isometric piecewise linear C^0-immersions. If, moreover, f_0 is a C^0-embedding, then the approximating piecewise linear isometries can be chosen among C^0-embeddings.*

Here the C^2-smoothness of an immersion is considered with respect to the (canonical) differential structure of the topological manifold M; this does not contradict the fact that the metric on M may have singularities.

1.5. Example. Consider a flat torus T^2 (obtained by identifying the opposite sides of a parallelogram). There is an obvious diffeomorphism f_0 mapping T^2 onto an ordinary rotation torus in $\mathbb{R}^3$. If the latter is sufficiently small, then f_0 is a short map. Theorem 1.4 shows that for any $\varepsilon > 0$, the flat torus T^2 admits an isometric piecewise linear embedding ε-close to the rotation torus.

1.6. More generally, every polyhedron M admits a C^2-smooth immersion into $\mathbb{R}^3$, and if M is either orientable or nonorientable but with nonempty boundary, then M admits also a C^2-embedding in $\mathbb{R}^3$. Using a contracting homothety, we can make the immersion (embedding) short with respect to the given polyhedral metric on M. Therefore, Theorem 1.4 implies the following statement.

1.7. Theorem. *Every polyhedron M admits an isometric piecewise linear C^0-immersion into $\mathbb{R}^3$. If M is orientable or has a nonempty boundary, then M admits an isometric piecewise linear C^0-embedding in $\mathbb{R}^3$.*

1.8. In [3], we proved a weaker version of Theorem 1.4 for embeddings in the case of orientable M; namely, it was proved that an embedding λf_0, and not f_0 itself, can be approximated, for a sufficiently small positive $\lambda = \lambda(M, f_0)$. Accordingly, [3] contains only a part of Theorem 1.7.

1.9. In passing, we fix a defect in the proof in [3]; we mean an incorrect statement in the first paragraph of §8 of [3]. To do this we replace §4 of [3] by a more appropriate construction.

1.10. *Remark.* Certainly, the requirement of C^2-smoothness imposed on the short immersion f_0 can be replaced by a weaker one. For example, it suffices to require that f_0 admit C^0-approximation by C^2-smooth (short) immersions f_{0i} (which is true, e.g., if f_0 is C^1-smooth).

1.11. Example. Let $0 < \lambda < 1$. If Q is the surface of a cube in $\mathbb{R}^3$, then any neighborhood of λQ contains an embedded polyhedral surface isometric to Q.

1.12. If a polyhedron M is the doubling (i.e., the result of identifying the boundaries of two copies) of a convex polygon, then, obviously, M can be immersed into $\mathbb{R}^3$ as a zero-volume convex polyhedron. Here (and also in 1.13 and in §9) by the volume of a polyhedron in $\mathbb{R}^3$ we mean the volume of the body bounded by it. On the other hand, by

Theorem 1.7, M also admits "realizations" by polyhedrons of nonzero volume embedded in $\mathbb{R}^3$. In general, such polyhedrons have much more faces than a natural triangulation of M. However, the example presented in §9 proves the following statement.

1.13. Theorem. *There exists a convex polyhedron P_1 of nonzero volume in $\mathbb{R}^3$ such that from the solid faces of P_1 one can construct a nonconvex polyhedron P_2 embedded in $\mathbb{R}^3$ and having the same combinatorial structure as P_1, but a larger volume,* $\operatorname{vol} P_2 > \operatorname{vol} P_1$.

1.14. *Remarks.* 1) Isometric piecewise linear maps of polyhedra (which, in general, are not C^0-embeddings) were also considered in [4, 5].

2) By the classical theorem of A. D. Aleksandrov [6], every polyhedral metric of nonnegative curvature on the sphere or on the plane can be realized by a convex surface (the realization is not a C^0-immersion only in the case of a convex dihedron-polygon). Complete polyhedral metrics of nonpositive curvature on the plane are embeddable in $\mathbb{R}^3$ as *saddle* surfaces, see [7].

3) Also, the methods of proofs of our Theorems 1.4 and 1.13 allow us to show that every convex polyhedron in $\mathbb{R}^3$ admits "linear bendings" in the sense of [8].

1.15. The layout of the paper. In §2, we describe the basic element of the construction that allows us to embed small acute triangles. In §3, we give the general outline of the proof in the orientable case and in the more complicated nonorientable case. In §4, we explain how to obtain a triangulation whose triangles are acute and as small as needed. We construct canonical embeddings of special kind for small neighborhoods of the vertices of the polyhedron. As a preparation for this, we first vary the initial map f_0 near the vertices. This local variation is described in §5. Yet another variation of f_0 outside some neighborhoods of the vertices makes the map very close to a conformal one; see §6. At this point, the proof in the orientable case is completed.

In the nonorientable case, we first cut the polyhedron to make it orientable, and immerse the resulting polyhedron (§7). Then we deform the immersion to join the edge of the cut (§8).

Theorem 1.13 is proved in §9.

§2. The basic element of the construction

2.1. In what follows, A_p ($p = 1, 2, 3$) are the vertices of an acute triangle T; a_p are the corresponding vertices of another acute triangle t; B and b are the centers of the circumscribed circles of T and t, and R and r are their radii; E_p and e_p denote the midpoints of the sides A_kA_l and a_ka_l (here p, k, l run over all the triples of different indices from 1 to 3). Finally, $H_p = BE_p$ and $h_p = be_p$ are the distances from B and b to the sides of T and t.

Assume that each side of T is longer than the corresponding side of t,

$$A_kA_l > a_ka_l \quad (k, l = 1, 2, 3; k \neq l). \tag{1}$$

Since both triangles are acute, we also have

$$R > r. \tag{2}$$

Consider a right prism in $\mathbb{R}^3$ with the base t (see Figure 1). Above the center b we find the point B' such that $B'a_p = R$. On the lateral faces of the prism we mark some points E'_p, E'_k, E'_l so that $a_kE'_pa_l$ be an equilateral broken line of length A_kA_l for each triple of different indices p, k, l. Under these assumptions, the following is true.

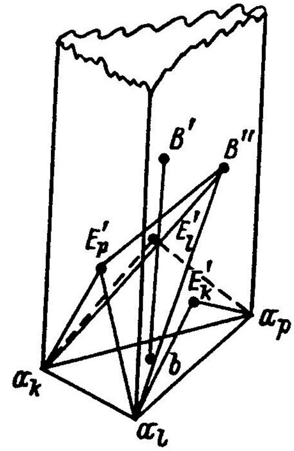

FIGURE 1

2.2. Lemma. *If, in addition to* (1), *the inequalities*

$$H_p > h_p \quad (p = 1, 2, 3) \tag{3}$$

hold, then the triangle T *can be isometrically piecewise linearly embedded in* $\mathbb{R}^3$ *as a "pleated" surface lying in the prism above* t *so that the boundary* $A_kA_lA_p$ *fits the broken line* $a_kE'_pa_lE'_ka_pE'_l$.

Proof. Consider the triangle part $A_kE_pA_lB$ of T separately. Having folded it along the line E_pB, we place it into the prism in such a way that the broken line $A_kE_pA_l$ coincide with $a_kE'_pa_l$. Then the segment E_pB takes the position E'_pB'' (see Figure 1). Inequality (3) shows that the segment E'_pB'' crosses the ray $\lambda = bB'$, i.e.,

$$E'_pB'' \cap \lambda \neq \varnothing. \tag{4}$$

Through the line a_ka_l we draw two planes P_1 and P_2 such that they cross the segment E'_pB'' and the points on E'_pB'' are ordered as follows:

$$E'_p \prec (E'_pB'' \cap P_2) \prec (E'_pB'' \cap P_1) \prec (E'_pB'' \cap \lambda) \prec B''.$$

We reflect in P_1 the part of the folded triangle $a_kE'_pa_lB''$ that contains B'', then reflect in P_2 the part of the result cut off by P_2, then again reflect in P_1, in P_2, etc. By continuity, the planes P_1 and P_2 can be chosen so that after an even number of reflections, B'' will hit the ray λ and hence exactly at the point B'. In this case, the triangle A_kA_lB turns out to be isometrically and piecewise linearly embedded in the prism above the triangle a_ka_lb.

Having performed this construction for each of the three parts A_kA_lB ($k \neq l = 1, 2, 3$), we obtain the required embedding of T. It is easily seen that, under a suitable choice of planes P_1 and P_2, the images of the triangles A_kA_lB have no common points except for their common sides A_kB. Such units will serve as the basic elements in the proofs of Theorems 1.4 and 1.13.

2.3. If two acute triangles T and t are similar, then condition (1) alone implies (3), and the above construction is realizable.

Now we assume that each angle φ of T satisfies the inequality

$$(5) \qquad 0 < \alpha < \varphi < \frac{\pi}{2} - \alpha$$

and that, instead of (1), the following stronger condition is fulfilled:

$$(6) \qquad C \cdot A_k A_l \geq a_k a_l, \quad C < 1.$$

If, in addition, T and t are sufficiently close to being similar (i.e., their sides are sufficiently close to being proportional), then (3) is valid, and it is possible to embed T above t by the above method.

2.4. The planes of the lateral faces of the prism based on t, together with the equilateral broken lines $a_k E'_p a_l$ on these faces, can be slightly turned around the lines $a_k a_l$ (independently of one another), but only as long as (4) is not violated. Under these conditions, it remains possible to realize the above embedding of T. The maximal angle of rotation of the planes admits a lower estimate in terms of the constants α and C from (5) and (6) and the accuracy of near-similarity of T and t.

2.5. In most cases, we shall construct the units described above simultaneously for a large number of pairs of triangles (T_i, t_i) for which inequalities (5) and (6) are satisfied with the same values of α and C, and the triangles of each pair are almost similar within sufficient accuracy.

2.6. For some other pairs (T_i, t_i), the value of C will be made so small that conditions (5) and (6) alone, without any near-similarity, imply (3) and guarantee the realizability of the construction.

§3. General outline the proof

Here we outline the proof of Theorem 1.4, first for the simplest case of a closed orientable polyhedron M (see 3.1–3.6). The case of an orientable manifold with boundary is discussed in 3.7, and the case of a nonorientable manifold in 3.8 and 3.9.

3.1. We start with varying the original $(1 - 5\varepsilon)$-short map f_0 in small neighborhoods of the vertices of M, so that in each neighborhood the new map has some standard form allowing us to construct a standard embedding near the corresponding vertex. The variation and the construction of the embedding will depend on the total angle θ around the vertex in question. They will be different for the cases $\theta < 2\pi$ and $\theta > 2\pi$; see §5. The map f_1 obtained by variation is $(1 - 2\varepsilon)$-short outside of some neighborhoods of the vertices with $\theta > 2\pi$.

3.2. We surround the vertices of M with small disjoint polygonal neighborhoods. The choice of the latter depends on f_1. Then we triangulate the complement of these neighborhoods in M into acute triangles. There exists $\alpha > 0$ such that inequality (5) is fulfilled for all the angles φ of the triangles of the triangulation; see 4.6.

The triangulation will be further refined (see §4), but in all refinements inequality (5) will be preserved for all the angles of all triangles, with the same α.

3.3. After that, the map f_1 is varied more substantially, this time outside certain neighborhoods of the vertices of M. This variation makes use of a familiar construction due to Kuiper [2], in which a finite number of C^2-smooth waves are successively superimposed on the surface of f_1. The resulting map f_2 remains C^2-smooth and $(1 - 2\varepsilon)$-short and becomes almost conformal outside of small neighborhoods of the vertices with $\theta > 2\pi$. The degree of "near-conformality" needed for further application of the arguments from 2.3 depends only on α and $C = 1 - \varepsilon$.

3.4. Then the triangulation chosen above is refined into sufficiently small triangles T_i. Joining the f_2-images of the nodes of the refined triangulation by segments (corresponding to the edges), we obtain some triangles t_i in $\mathbb{R}^3$. Since the map f_2 is almost conformal, and the triangulation is sufficiently fine, we may assume that outside small neighborhoods of the vertices, the t_i are as close to being similar to the corresponding triangles T_i as we need. Moreover, all the dihedral angles formed by the adjacent pairs of triangles t_i are uniformly close to π, provided the triangulation is fine enough. (This allows us to use the arguments from 2.4.)

3.5. We fix one of the two sides of the polyhedral surface formed by the triangles t_i and regard the bisectors of the bihedral angles between each given t_i and its adjacent triangle as the planes of the lateral faces of a truncated pyramid (possibly degenerated into a prism) having t_i as the base and placed above that fixed side of the surface. Then we embed each triangle T_i into the corresponding pyramid by using the construction described in §2.

For the triangles lying in the region of "near-conformality" of f_2 the above embedding is constructed with the help of the arguments from 2.3 and 2.4, and for those lying near the vertices with $\theta < 2\pi$ the arguments from 2.6 are applicable. In the vicinity of the vertices with $\theta > 2\pi$, another construction of embedding will be used; see 5.6.

3.6. Since the surface f_2 is C^2-smooth and compact, the embedding units (i.e., the images of the T_i) situated above close (in the structure of the surface) triangles t_i are disjoint if the triangulation is sufficiently fine. Thus, the entire construction yields a piecewise linear, isometric immersion of M into $\mathbb{R}^3$, which is an embedding if f_0 is. For more details about the way of avoiding "self-intersections" we refer the reader to §5 of [3].

Obviously, the entire construction can be performed within any prescribed C^0-neighborhood of f_0.

3.7. If M is an orientable polyhedron with boundary, the construction of embedding is almost the same. The only difference is that the procedures used in the neighborhoods of the vertices with $\theta < 2\pi$ ($\theta > 2\pi$) are also used at the corner points of the boundary of M with $\theta \leq \gamma$ ($\theta > \gamma$), where γ is the angle formed by the surface f_0 at the corresponding boundary point.

Note that in order to prove Theorem 1.7 alone, one need not consider an orientable polyhedron M with nonempty boundary, since it suffices to pass to the doubling of M. Similarly, in the nonorientable case, the corner points can be removed by attaching to the boundary a strip without corners on the exterior border.

The realizability of the plan presented in 3.1–3.7 in the case of *orientable* M is justified in §4–6 below.

3.8. Now we turn to the case of a nonorientable polyhedron M, where the construction needs to be supplemented substantially.

Every nonorientable compact 2-manifold M is homeomorphic either to the projective plane with holes and handles, or to the Klein bottle with holes and handles. (There may be no holes or handles.) M can be made orientable by cutting along a suitable disorienting cycle in the former case, and along two such disjoint cycles in the latter case.

We start the construction with picking such a cut (respectively, two cuts) on M. For definiteness, below we discuss the case of a single cut. We may assume that this cut is a simple C^2-smooth closed curve Γ passing through no vertices and no boundary points of M.

3.9. Cut along Γ, the polyhedron is orientable. Unlike the polyhedra with boundary, it contains a curved boundary component $\mathcal{L}$ corresponding to passing twice along Γ.

Along $\mathcal{L}$ we construct a triangulation of a special kind; see §7. Every refinement of it leads to replacement of this triangulation (in a certain standard way) by the corresponding broken line inscribed into $\mathcal{L}$, so that actually we cut the polyhedron not along Γ but along a broken line L inscribed into Γ.

After that, we choose one of the two sides of the cut surface f_2 and, applying procedures similar to those of 3.3–3.7, construct an isometric piecewise linear immersion (or embedding) of the polyhedron M cut along L. The embedding units formed by the triangles T_i located on different sides of L are placed on different sides of the surface f_2. Therefore, this embedding of the polyhedron M cut along L does not yield the required immersion of the entire polyhedron M because of the gap along L. To get rid of that gap, a strip of f_2 adjacent to $f_2(L)$ is deformed in a special way. This allows us to "patch" the gap; see §§7, 8.

§4. The triangulation and its refinement

4.1. In §§2, 3 of [3] we proved that every polyhedron M can be split into acute triangles with entire sides in common.

The idea of the proof is as follows. First, the polyhedron is divided in a special way into acute triangles, possibly adjoining one another not along entire sides. Then this subdivision is refined, and some vertices of the refined subdivision are slightly moved along the edges of the original one to form the required triangulation. The possibility of finding a suitable subdivision and suitable shifts of the vertices is provided by the well-known theorem about uniformly good approximation of a collection of real numbers by rational ones with common (arbitrarilly large) denominator (see, e.g., [9, Chapter 1, §5]).

The construction of such an acute triangulation allows us to ensure some additional properties. In particular, let M have a boundary, and let several portions of the boundary be equilateral broken lines (each portion has its own length of segments). The method given in [3] allows us to assume that the acute triangulation Φ of M is chosen in such a way that within each of those portions all segments are divided by Φ into the same number of equal parts.

4.2. For our purposes, before constructing the acute triangulation Φ, we encircle every vertex A of M with total angle $\theta < 2\pi$ by a small regular hexagon $Q(A)$ composed of six isosceles triangles with apical angle $\theta/6$.

4.3. The map f_0 is assumed to be C^2-smooth up to the boundary of the polyhedron. Therefore, the f_0-image of a neighborhood of a corner point A resembles a sector-like portion of a C^2-smooth surface, bounded by two smooth curves that form an angle $\lambda < 2\pi$. If A is a corner point of M with total angle $\theta \le \lambda$, then, as in 4.2, we draw a polygon $Q(A)$ composed of six isosceles triangles filling the sector of angle θ at A.

4.4. Before choosing the acute triangulation Φ, we vary f_0 near the vertices B with $\theta > 2\pi$ (and the corner points with $\theta > \lambda$) as described below in 5.4. After the variation, the resulting map f_1 takes some circular neighborhood $V(\rho_4)$ of every such vertex B onto a plane disk (or sector). Moreover, within $V(\rho_4)$ the map f_1 is isometric on the radial segments and is uniformly contracting on circles centered at B (so, f_1 is not C-short in this region).

We split the boundary of $V(\rho_4)$ in M into $N > 10^4\theta/2\pi$ equal arcs of length, say, 2δ. Each arc is the base of an isosceles "triangle" in the exterior of $V(\rho_4)$ in M, with lateral

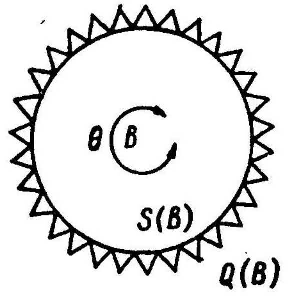

FIGURE 2

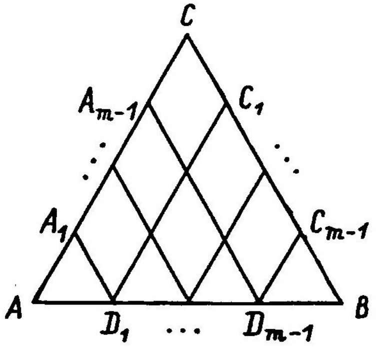

FIGURE 3

sides of length 2δ. As a result, we obtain a "cogged" polygon $Q(B)$ containing $V(\rho_4)$ (see Figure 2).

A similar construction is performed near each corner point B with $\theta > \lambda$. In that case, $V(\rho_4)$ is a sector, $Q(B)$ is a "cogged sector", and $N > 10^4\theta/\lambda$.

4.5. All polygons $Q(A)$ and $Q(B)$ can be chosen so small that they are disjoint and none of them contains "extraneous" vertices or boundary points.

4.6. For the polyhedron M with all the neighborhoods $Q(A)$ and $Q(B)$ excluded, we construct an acute triangulation Φ as outlined in 4.1.

Since the total number of triangles from Φ is finite, there exists $\alpha > 0$ such that inequality (5) is satisfied for all the angles φ of the triangles from Φ and of the triangles constituting the polygons $Q(A)$.

4.7. To refine Φ, we take an arbitrarily large n, split each edge into n equal segments and, accordingly, split each triangle into n^2 triangles similar to it. Then the triangles of the refined triangulation that adjoin $Q(A)$ or $Q(B)$ from outside split each side of $Q(A)$ (or $Q(B)$) into equal segments. Therefore, the refined triangulation can be naturally extended to $Q(A)$ by splitting each of the six triangle components of $Q(A)$ into similar triangles.

For any n, all the angles φ of the triangles of the refined triangulation satisfy inequality (5), and the constant α remains unchanged.

4.8. The extension of the refinement to the cogs of the polygons $Q(B)$ is constructed in a somewhat different way.

Consider the cog ABC shown in Figure 3. The sides AC and CB are segments of length 2δ, and the "base" AB is a circle arc of length 2δ. Since the angle subtended by the arc is less than 10^{-4}, the shape of the cog ABC is close to an equilateral triangle.

Suppose that the refinement of Φ splits each of the sides AC and CB into m equal segments by points $A_1, \dots, A_{m-1}$ and $C_1, \dots, C_{m-1}$; see Figure 3. We split the arc AB into m equal arcs by points $D_1, \dots, D_{m-1}$ and then draw the segments $D_1A_1, \dots, D_{m-1}A_{m-1}$ and $D_1C_1, \dots, D_{m-1}C_{m-1}$. The blocks of the resulting partition are close to rhombuses. Next, we divide each block by its diagonal (the one whose direction is close to AB) and draw the broken line joining the points $A, D_1, \dots, D_{m-1}, B$. All the resulting triangles form the extension of the refined triangulation.

If the triangulation is sufficiently fine, then all the angles of the above extension of the refined triangulation satisfy (5) with the same α as before.

§5. The first variation of the map and embedding of neighborhoods of the vertices

5.1. In a plane sector $K(\theta)$ of angle θ we introduce polar coordinates (φ, ρ), $0 \le \varphi \le \theta$, $\rho > 0$, and, in a plane sector $K(\lambda)$, similar coordinates (ψ, r), $0 \le \psi \le \lambda$, $r > 0$. The map $K(\theta) \to K(\lambda)$ defined by

$$\psi = \frac{\lambda}{\theta}\varphi, \quad r = a\rho^{\lambda/\theta}, \tag{7}$$

is conformal for any positive a. (For $\lambda = 2\pi$ this is a conformal map of a cone with total angle θ onto the plane.) We call (7) the *standard conformal map*.

At a point (φ, ρ), the ratio of the linear element ds in $K(\lambda)$ to the linear element $d\sigma$ in $K(\theta)$ is given by the formula

$$\frac{ds}{d\sigma} = \frac{dr}{d\rho} = a\frac{\lambda}{\theta}\rho^{\lambda/\theta - 1}. \tag{8}$$

If $\theta < \lambda$ and ρ is small, then this map is short. Moreover, for any $C \in (0,1)$ there exists ρ_0 such that this map is C-short in the disk $\rho < \rho_0$.

5.2. Let A be a vertex of M with total angle $\theta < 2\pi$.

Let P be the tangent plane to our C^2-smooth surface f_0 at the point $f_0(A)$, and let $U(\rho_1)$ be a circular neighborhood of A with sufficiently small radius $\rho_1 > 0$. The original $(1-5\varepsilon)$-short map f_0 can be varied in $U(\rho_1)$ in such a way that the resulting map remain C^2-smooth, be $(1-4\varepsilon)$-short, and take some circular neighborhood $U(\rho_2)$ (where $0 < \rho_2 < \rho_1$) of A to a disk in P centered at $f_0(A)$. Additionally, we can vary the map in $U(\rho_2)$ so that it become the standard conformal map of some circular neighborhood $U(\rho_3)$ of the vertex A $(0 < \rho_3 < \rho_2)$ onto a plane disk, remaining C^2-smooth and $(1-3\varepsilon)$-short in $U(\rho_2)$.

5.3. Refining the triangulation Φ, we split the polygon $Q(A)$ into small isosceles triangles with apical angle $\theta/6$. Inside the circular neighborhood $U(\rho_3)$, there is a neighborhood $U(\rho_4)$, $0 < \rho_4 < \rho_3$, in which the above standard conformal map satisfies the following two conditions. First, the vertices of the triangles T_j are mapped to vertices of acute triangles t_j in the plane P. Second, the coefficient of contraction is so large that the arguments from 2.6 allow us to embed every $T_j \subset U(\rho_4)$ above the corresponding triangle t_j as described in Lemma 2.2. The embeddings of the triangles T_j adjacent to A form a piecewise linear isometric embedding of a certain neighborhood (in M) of A.

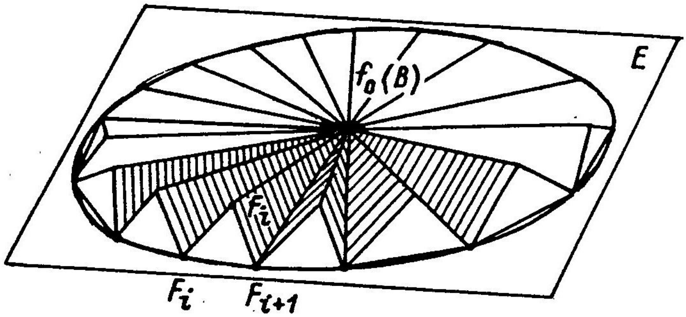

FIGURE 4

5.4. A similar construction (after a due refinement of Φ) yields an embedding of some neighborhoods of the corner points A at which $\theta < \lambda$. In this case, the neighborhoods $U(\rho_i)$ are sectors with angle θ, and $U(\rho_3)$ is mapped onto a plane sector of angle λ.

For a corner point with $\theta = \lambda$, the map (7) reduces to a homothety of ratio $a \leq 1-3\varepsilon$, and the construction of the embedding is even simpler.

5.5. Let B be a vertex of M with total angle $\theta > 2\pi$.

As in 5.2, the map f_0 can be varied in a small circular neighborhood $V(\rho_1)$ of B so that the resulting map remain C^2-smooth, be $(1-4\varepsilon)$-short, and take a neighborhood $V(\rho_2)$ of B, $0 < \rho_2 < \rho_1$, to a disk centered at $f_0(B)$ in the plane P tangent to the surface f_0 at $f_0(B)$.

Additionally, we vary the map in $V(\rho_2)$ to make it satisfy the following conditions:

1) In a smaller closed circular neighborhood $V(\rho_4)$ centered at B, the map is isometric on the radial segments and uniformly contracting with the coefficient $\theta/2\pi$ on the circles centered at B.

2) In a certain annular neighborhood $V(\rho_3) \setminus V(\rho_4)$, where $0 < \rho_4 < \rho_3 < \rho_2$, it is the standard conformal map with the coefficient of contraction equal to $\theta/2\pi$ at the boundary of $V(\rho_4)$. This can be achieved by choosing an appropriate constant a in (7).

3) Outside $V(\rho_4)$ this map remains C^2-smooth and $(1-3\varepsilon)$-short. (We may assume that $\varepsilon > 0$ satisfies $1-3\varepsilon > 2\pi/\theta$.) At the boundary of $V(\rho_4)$ the resulting map f_1 is only C^1-smooth.

5.6. Any refinement of the triangulation Φ results in splitting the boundary of the circular neighborhood $V(\rho_4)$ into some number (say, N_m) of arcs of equal length. Let $F_1, \dots, F_{N_m}$ be the splitting points, and let E_i denote the midpoint of the chord F_iF_{i+1}. Then the polygon $V(B) = F_1 \dots F_{N_m}$ is isometrically and piecewise linearly embedded above the plane P as a radially crimped surface shown in Figure 4.

If Φ is sufficiently fine, then every equilateral broken line $F_iE_iF_{i+1}$ lies in a plane almost perpendicular to P. Therefore, it is possible (by using the arguments from 2.4) to bring $F_iE_iF_{i+1}$ into coincidence with the side F_iF_{i+1} of an adjacent triangle T_j of the refined triangulation embedded above P by the method of Lemma 2.2.

Neighborhoods of the corner points B with $\theta > \lambda$ are embedded in a similar way.

§6. THE SECOND VARIATION OF THE MAP

6.1. The map f_1 is already conformal in the neighborhoods $U(\rho_3)$ of the vertices A with $\theta < 2\pi$ and of the corner points A with $\theta \leq \lambda$. It is also conformal in the annular neighborhoods $V(\rho_3) \setminus V(\rho_4)$ of the vertices B with $\theta > 2\pi$ and of the corner points with $\theta > \lambda$. Further variation of f_1 is performed outside of all the neighborhoods $U(\rho_3)$ and $V(\rho_3)$.

6.2. For each vertex (or corner point), we take a C^∞-smooth function $\psi : (0 < \rho \le \rho_3) \to \mathbb{R}$ such that

$$\psi(\rho) = \begin{cases} 1 & \text{if } 0 < \rho \le \rho_4, \\ \text{some monotone function of } \rho & \text{if } \rho_4 \le \rho \le \rho_3, \\ \frac{1-2\varepsilon}{C(\rho_3)} & \text{if } \rho = \rho_3, \end{cases}$$
$$\psi'(\rho_3) = \psi''(\rho_3) = 0.$$

Here the values ρ_3 and ρ_4 correspond to the given vertex (or corner point), and $C(\rho_3)$ is the coefficient of contraction of the metric under f_1 at the border of $U(\rho_3)$ or $V(\rho_3)$.

Let ds be the linear element of the (flat) metric of M off the vertices, and ds_1 the corresponding linear element of the metric on the surface f_1. We introduce a new metric on M by the formulas

$$ds^* = \begin{cases} \psi\, ds_1 & \text{in } U(\rho_3) \text{ and } V(\rho_3), \\ (1-2\varepsilon)ds & \text{outside of these neighborhoods.} \end{cases}$$

Everywhere outside of the vertices and the neighborhoods $V(\rho_4)$, this metric is conformally equivalent to the metric ds and is $(1-2\varepsilon)$-short with respect to ds.

6.3. Outside of $U(\rho_3)$ and $V(\rho_3)$, the map $f_1 \colon M \to \mathbb{R}^3$ is C^2-smooth and short with respect to ds^*. For any $\Delta > 0$, using a standard construction due to Kuiper [1], we can vary f_1 outside of all $U(\rho_3)$ and $V(\rho_3)$ in such a way that the metric ds_2 induced by the resulting C^2-smooth map f_2 be Δ-close to ds^*. The latter condition means that for all directions at every point we have

$$|ds_2 - ds^*| \le \Delta \cdot ds^*.$$

Moreover, f_2 is an immersion or embedding if so are f_0 and f_1.

Outside of the regions $V(\rho_4)$, the map f_2 remains $(1-\varepsilon)$-short with respect to ds.

If we choose $\Delta > 0$ sufficiently small, then, outside of the regions $U(\rho_4/2)$ and the polygons $V(B)$, every triangle T_i of any sufficiently fine triangulation Φ is almost similar (with required accuracy) to the triangle $t_i \subset \mathbb{R}^3$ whose vertices are the f_2-images of those of T_i.

6.4. The principal curvatures of the surface f_2 are uniformly bounded off $U(\rho_4)$ and $V(\rho_4)$. Therefore, if the triangulation is sufficiently fine, then the dihedral angles formed by the adjacent uniformly acute triangles t_i are close to π.

Again, if the triangulation is sufficiently fine, then the polygons $V(B)$ can be embedded by radially crimped surfaces as in 5.6 and Figure 4, the triangles T_i lying inside $U(\rho_4)$ can be embedded above t_i as in 2.6, and the remaining T_i, which lie outside of the neighborhoods $V(B)$ and $U(\rho_4/2)$, can be embedded above t_i as in 2.5.

Since the dihedral angles formed by the pairs of adjacent triangles t_i are close to π, and the planes $F_iE_iF_{i+1}$ (see 5.6) are almost perpendicular to the plane P, all those embeddings can be joined together as explained in 2.4.

In addition, the fineness of a triangulation guarantees that the isometric piecewise linear map $M \to \mathbb{R}^3$ obtained in this way remains an immersion (or embedding).

This completes the proof of Theorem 1.4 in the case where M is orientable.

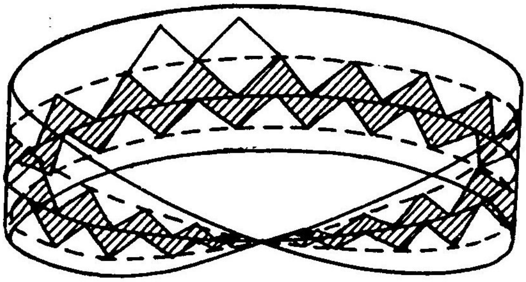

FIGURE 5

§7. THE CUT AND ITS STRIP

7.1. Let nonorientable polyhedron M be cut along a disorienting C^∞-cycle Γ (or two such cycles) so that the cut manifold M be orientable. We split the cycle Γ into an odd number of intervals of equal length so small that the rotation, i.e., the integral (with respect to the length) of the absolute value of the geodesic curvature, of each interval, be less than, say, 10^{-10}. Let δ be the common length of such intervals.

Each pair of adjacent intervals is a slightly curved arc of length 2δ. As on bases, on these arcs we draw triangle cogs (by turns to the right and to the left from Γ) with lateral sides of length 2δ. The cogs are shown by hatching in Figure 5.

7.2. Then we join the apices of the adjacent cogs lying on the same side of Γ by segments shown in Figure 5 by dashed lines. The lengths of these segments slightly differ from one another (although they all are close to 2δ); therefore, additionally, on each of the segments as on a base we draw an external isosceles triangle with lateral sides of length 2δ. Thus, Γ is surrounded by a cogged strip $Q(\Gamma)$ whose boundary is formed by segments of length 2δ.

The cutset and its strip $Q(\Gamma)$ can be chosen so that $Q(\Gamma)$ contain no vertices or boundary points of M and no points of the previously constructed polygons $Q(A)$ and $Q(B)$.

Now we construct an acute triangulation Φ for the polyhedron M excluding not only all the $Q(A)$ and $Q(B)$ but also the strip (or two strips) $Q(\Gamma)$. Moreover, Φ should split all the boundary segments of $Q(\Gamma)$ into one and the same number of equal parts. Let this number be $2^m(2q+1)$. At the expense of an additional refinement we may assume that $q > m$.

7.3. The triangulation Φ is extended to the strip $Q(\Gamma)$ in such a way that each of the above-mentioned arcs of length 2δ on Γ be split into the *odd* number $3^m(2q+1)$ of parts of equal length.

This is done as follows. First, we extend Φ to the external triangle cogs of $Q(\Gamma)$ in the usual way; see Figure 5. Then the upper bases of the next row of triangles (only their apices lie on Γ) are split into $2^m(2q+1)$ segments each. To these and the remaining triangles we extend only the splitting of each side into $2q+1$ equal parts, and split each triangle into $(2q+1)^2$ ones. For the "triangles" with curvilinear bases on Γ this is done as described in 4.8.

Now Γ is surrounded by a strip of $2q+1$ rows of triangles (the external cogs are excluded). In the exterior row, the exterior base of each triangle is divided into 2^m parts by the previous extension of Φ. For each of those triangles we bisect the base and trisect each of the lateral sides, and then subdivide the triangle as in Figure 6. After that, the splitting of the sides into 3 parts (and of each triangle into 9 parts) is extended to

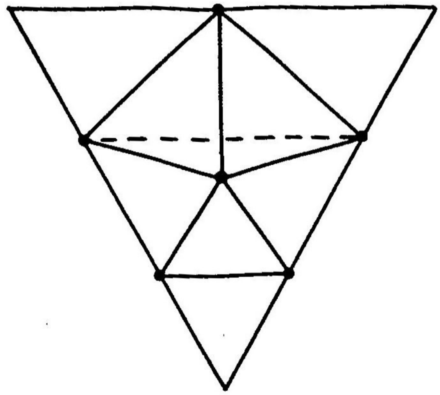

FIGURE 6

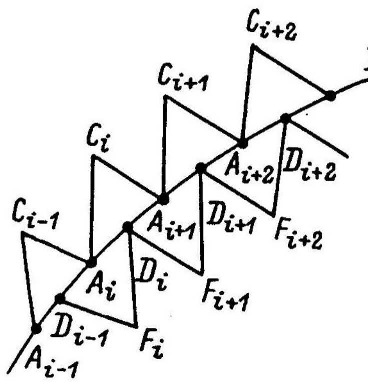

FIGURE 7

the strip up to Γ. Thus, the initial splitting of the exterior bases into 2^m parts yields a splitting of the bases of the next generation of triangles into 2^{m-1} parts each, and the "next exterior row" itself becomes "three times as narrow".

Having repeated this construction m times (i.e., through m more and more narrow rows), we obtain the required extension of Φ down to Γ.

Only after this is the constant α in (5) chosen.

7.4. The triangulation Φ may be further refined arbitrarily, but we agree that all refinements split the sides of the triangles into an odd number of parts.

In any triangulation so refined, Γ is adjoined by the "triangles" $A_{i-1}C_{i-1}A_i$ and $D_{i-1}F_iD_i$ (see Figure 7) based on some curvilinear intervals on Γ. We replace the cutset Γ with the broken line $A_{i-1}D_{i-1}A_iD_i \dots$.

Each of the quadrangles $A_{i-1}C_{i-1}A_iD_{i-1}$ (and of the similar ones $D_{i-1}F_iD_iA_i$) is close in shape to an equilateral triangle. At the expense of refining the triangulation, the angles $\angle A_{i-1}D_{i-1}A_i$ can be made arbitrarily close to π.

7.5. If the triangulation is sufficiently fine, we can embed all its triangles T_j (on one side from the cut surface $f_2(M)$) above the corresponding triangles t_j whose vertices are the f_2-images of those of T_j. The triangles $a_{i-1}c_{i-1}a_i$, where $a_j = f_2(A_j)$ and $c_j = f_2(C_j)$, make an exception, and above such a triangle we must embed the quadrangle $A_{i-1}C_{i-1}A_iD_{i-1}$, and not the triangle $A_{i-1}C_{i-1}A_i$. Since the angles $\angle A_{i-1}D_{i-1}A_i$ are close to π this can be done as follows.

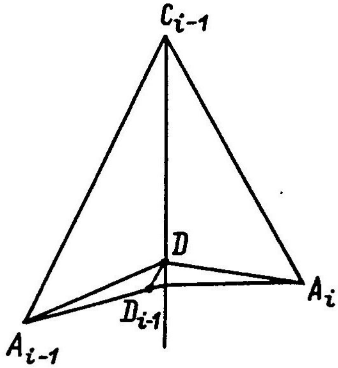

FIGURE 8

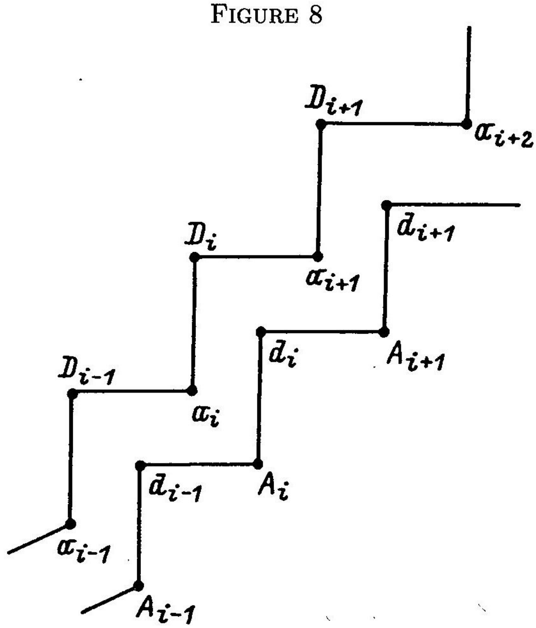

FIGURE 9

We construct a truncated pyramid based on the triangle $a_{i-1}c_{i-1}a_i$. The planes of the lateral faces of this piramid adjoining the edges $a_{i-1}c_{i-1}$ and a_ic_{i-1} are the bisectors of the dihedral angles formed by $a_{i-1}c_{i-1}a_i$ and the adjacent triangles t_j. The plane of the face adjoining $a_{i-1}a_i$ is parallel to the normal of the surface $f_2(M)$ at the point $d_{i-1} = f_2(D_{i-1})$. Next, on the bisector of the angle C_{i-1} of the quadrangle $A_{i-1}C_{i-1}A_iD_{i-1}$ we mark a point D so that the segments DA_{i-1} and DA_i be still inside the quadrangle, but the angle $\angle A_{i-1}DA_i$ be very close to π; see Figure 8.

Then we single out the quadrangle $A_{i-1}DA_iD_{i-1}$, fold it along $D_{i-1}D$, and embed it into the pyramid in such a way that the points A_{i-1} and A_i coincide with a_{i-1} and a_i, the broken line $A_{i-1}D_{i-1}A_i$ lie in the lateral face, and the point D be inside the pyramid. The remaining part $A_{i-1}C_{i-1}A_iD$, having already exactly equal sides $A_{i-1}D = DA_i$, can be embedded with the help of the construction described in §2.

The neighborhoods of the vertices and the corner points of M are embedded as in §5.

As a result, the polyhedron M cut along the broken line $L = A_{i-1}D_{i-1}A_iD_i\dots$ is immersed (or embedded) into $\mathbb{R}^3$, but the two sides of the broken line $a_{i-1}d_{i-1}a_id_i\dots$

do not fit together. The "gates" $a_{i-1}D_{i-1}a_i$, $d_{i-1}A_id_i, \dots$ shown in Figure 9 are almost orthogonal to the surface $f_2(M)$ but go in opposite directions relative to it. So, the above immersion of the M cut along L still keeps it cut.

§8. Joining the cutset

8.1. Let t and τ denote the arc-length parameters on $\Gamma \subset M$ and on $f_2(\Gamma)$, respectively. The function $\varphi(t) = \frac{d\tau}{dt}$ is C^1-smooth and positive. Moreover, $\varphi < 1 - \varepsilon$, because f_2 is $(1-\varepsilon)$-short.

With every point $f_2(\Gamma(t))$ we associate the plane $P(t)$ that contains the normal of the surface f_2 at that point and is tangent to $f_2(\Gamma)$.

If, under a refinement of the triangulation, each segment $A_{i-1}A_i$ of Γ of length 2δ is split into N parts, then for N large every triangle $\Delta = a_{i-1}D_{i-1}a_i$ lies in a plane whose direction is close to that of $P(t)$, where t is the value of parameter corresponding to D_{i-1} (see Figure 8). The shape of Δ is close to that of the isosceles triangle $\Delta' = a'_{i-1}D'_{i-1}a'_i$ with sides

$$a'_{i-1}D'_{i-1} = a'_iD'_{i-1} = \frac{\delta}{N}, \qquad a'_{i-1}a'_i = \frac{2\delta}{N}\varphi(t).$$

The altitude h of Δ' drawn from D'_{i-1} is of length

$$h = \sqrt{1-\varphi^2(t)}\frac{\delta}{N}.$$

Therefore, as the triangulation grows finer, the direction of each segment $D_{i-1}d_{i-1}$ in Figure 8 approaches the normal to the surface f_2 at d_{i-1}, and the length of each segment becomes close to $\sqrt{1-\varphi^2(t)}\frac{\delta}{N}$.

8.2. The cut surface f_2 contains the twice-passed cycle $f_2(\Gamma)$ as a boundary component. Any refinment of our triangulation gives us a polyhedral surface inscribed into the cut surface f_2 and bounded by the broken line $l = \dots a_{i-1}a_ia_{i+1}\dots d_{i-1}d_id_{i+1}\dots$. On this orientable polyhedral surface, the broken line l is adjoined (from one side only) by a strip composed of many rows of almost equilateral triangles t_j, which are the images of the triangles lying in the strip $Q(\Gamma)$ cut along the broken line $L = \dots A_{i-1}D_{i-1}A_iD_i\dots$.

We shall vary f_2 in a C^2-smooth way within the strip $Q(\Gamma)$ cut along Γ. The regions of the surface f_2 located on two different sides of $f_2(\Gamma)$ are varied independently and lose their fitting together at $f_2(\Gamma)$. In the process, the polyhedral surface composed of the triangles t_j, also undergoes deformation. Such a variation is said to be *admissible* if above all the triangles t_j it remains possible to construct coordinated embeddings of the corresponding triangles T_j (and the quadrangles of the form $A_{i-1}C_{i-1}A_iD_{i-1}$).

8.3. If the triangulation is sufficiently fine, then there exists an admissible variation that shifts each boundary point $f_2(\Gamma(t))$ by a distance of $\frac{1}{2}\sqrt{1-\varphi^2(t)}\frac{\delta}{N}$ along the normal at $f_2(\Gamma(t))$, in the direction opposite to the side of the cut surface where the triangles T_j are constructed (on the given side of $f_2(\Gamma)$).

After such a variation, the pair of points d_{i-1} and d_i becomes close to the pair D_{i-1}, D_i (see Figure 8).

If the triangulation is sufficiently fine, then it is possible to bring each d_i into coincidence with the corresponding point D_i by an additional admissible variation.

After that, using the remaining flexibility of the embeddindgs of $D_{i-1}F_iD_iA_i$ (see 2.4) we can slightly turn each of the triangles $d_{i-1}A_id_i$, making the entire contour $\dots D_{i-1}a_iD_ia_{i+1}D_{i+1}\dots$ coincide with $\dots d_{i-1}A_id_iA_{i+1}d_{i+1}\dots$.

This completes the proof of Theorem 1.4 in the case where M is nonorientable.

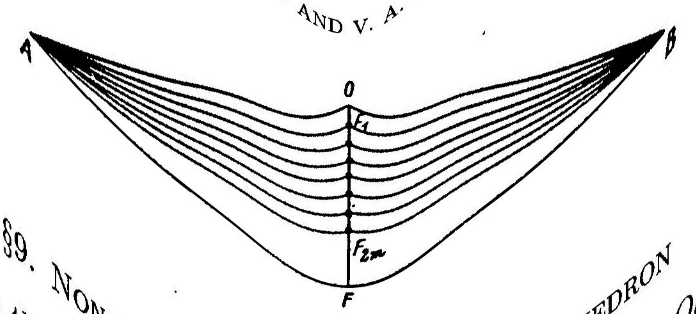

FIGURE 10

§9. Nonconvex inflation of a convex polyhedron

9.1. Let Q be the twice-covered regular hexagon $P_1 \cdots P_6$ centered at O_0. The pairs of segments O_0P_i divide Q into 12 equilateral triangles. Splitting each of them into n^2 congruent equilateral triangles, we obtain a triangulation of Q into triangles T_j.

Let S be a sphere of radius r with poles B_1 and B_2 and center O'. We divide the equator of S into 6 equal parts by the points $A_1, \dots, A_6$, split each plane triangle $B_kA_iA_{i+1} (k = 1, 2)$ into n^2 congruent triangles, and project the edges of this triangulation onto S from O'. This gives us a triangulation of S into spherical triangles $a_jb_jc_j$. The rays passing from O' through the plane triangle t_j with the same vertices $a_jb_jc_j$ fill a truncated pyramid based on t_j.

This triangulation of the sphere into the triangles $a_jb_jc_j$ corresponds to our triangulation of Q. If n is sufficiently large and S is small compared to Q, then for each j the construction from §2 allows us to embed the triangle T_j above the corresponding t_j. Thus, we obtain a polyhedral isometric embedding $Q \to \mathbb{R}^3$ yielding a polyhedron of nonzero volume.

9.2. In the above example, it is still possible to embed the triangles T_j above the triangles t_j provided the shape of T_j is slightly varied. Therefore, by shifting a little the nodes of the triangulation of Q, we can change Q into a new, "almost plane" convex polyhedron Q_1 of very small volume such that all (slightly deformed) triangles T_j are the true faces of Q_1. It remains possible to increase the volume of Q_1 under another (nonconvex) isometric embedding of it in $\mathbb{R}^3$.

9.3. The embedding of a single face T_j of Q_1 into a pyramid based on t_j by the method of §2 involves three steps. First, the triangle $T_j = ABC$ is divided into the isosceles triangles OAB, OBC, and OCA, where O is the center of the circumscribed circle. Second, each of the triangles is divided into a series of thin triangles; see the triangles $AF_iF_{i+1} = BF_iF_{i+1}$ in Figure 10.

After that, the construction described in §2 is realized by folding along the segments OF_1, AF_i, F_iF_{i+1}, F_iB, $F_{2m}F$.

We can replace the polyhedron Q_1 with another convex polyhedron Q_2 obtained from Q_1 by raising the center O of each face $T_j = ABC$ a little above the plane of T_j. The faces of Q_2 are isosceles triangles close to OAB, OBC and OCA.

Finally, for every face OAB of Q_2 it is possible to choose the points $F_1, \dots, F_{2m}$ and to raise the points F and F_i a little above the plane OAB so that, first, the triangles AOF_1, BOF_1, AF_iF_{i+1}, BF_iF_{i+1}, $AF_{2m}F$, $BF_{2m}F$ (see Figure 10) be true faces of the convex hull Q_3 of the vertices of Q_2 and all the raised points F and F_i from all the faces of Q_2, and second, precisely these triangles realize the embeddings of §2.

The example presented by the polyhedron Q_3 proves Theorem 1.13.

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