AN ODYSSEY INTO LOCAL REFINEMENT AND MULTILEVEL PRECONDITIONING II: STABILIZING HIERARCHICAL BASIS METHODS *

BURAK AKSOYLU† AND MICHAEL HOLST‡

Abstract. In this article, we examine the wavelet modified (or *stabilized*) hierarchical basis (WHB) methods of Vassilevski and Wang, and extend their original quasiuniformity-based framework and results to several types of local 2D and 3D red-green and red refinement procedures. The concept of a stable Riesz basis plays a critical role in the original work on WHB, and in the design of efficient multilevel preconditioners in general. We carefully examine the impact of local adaptive mesh refinement on Riesz bases and matrix conditioning. In the analysis of WHB methods, a critical first step is to establish that the BPX preconditioner is optimal for the refinement procedures under consideration, and to develop a number of supporting results for the BPX preconditioner. Therefore, the first article in this series was devoted to extending the results of Dahmen and Kunoth on the optimality of BPX for a certain type of 2D local refinement to additional types of 2D and 3D local refinement procedures. These results from the first article, together with the local refinement extension of the WHB analysis framework presented here, allow us to establish optimality of WHB preconditioner on several types of locally refined meshes in both 2D and 3D. More precisely: with PDE coefficients in C^1, we establish optimality for the multiplicative WHB method on locally refined meshes in both 2D and 3D. Without such smoothness assumptions, we show that the early suboptimal results can also be extended to locally refined meshes. With the minimal smoothness assumption that PDE coefficients are in L_∞, we establish optimality for additive WHB on the same classes of locally refined meshes in both 2D and 3D. An interesting implication of the optimality of WHB preconditioner is the *a priori* H^1-stability of the L_2-projection. The existing *a posteriori* approaches in the literature dictate a reconstruction of the mesh if such conditions cannot be satisfied. The proof techniques employed throughout the paper allow extension of the optimality results, the H^1-stability of L_2-projection results, and the various supporting results to arbitrary spatial dimension $d \geq 1$.

Key words. finite element approximation theory, multilevel preconditioning, hierarchical bases, wavelets, two and three dimensions, local mesh refinement, red and red-green refinement.

AMS subject classifications. 65M55, 65N55, 65N22, 65F10

1. Introduction.

In this article, we analyze the impact of local adaptive mesh refinement on the stability of multilevel finite element spaces and on the optimality (linear space and time complexity) of multilevel preconditioners. Adaptive refinement techniques have become a crucial tool for many applications, and access to optimal or near-optimal multilevel preconditioners for locally refined mesh situations is of primary concern to computational scientists. The preconditioners which can be expected to have somewhat favorable space and time complexity in such local refinement scenarios are the hierarchical basis (HB) method, the Bramble-Pasciak-Xu (BPX) preconditioner, and the wavelet modified (or stabilized) hierarchical basis (WHB) method. While there are optimality results for both the BPX and WHB preconditioners in the literature, these are primarily for quasiuniform meshes and/or two

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† Department of Computer Science, California Institute of Technology, Pasadena, CA 91125, USA (baksoylu@cs.caltech.edu).

‡ Department of Mathematics, University of California at San Diego, La Jolla, CA 92093, USA (mholst@math.ucsd.edu).

space dimensions (with some exceptions noted below). In particular, there are few hard results in the literature on the optimality of these methods for various realistic local mesh refinement hierarchies, especially in three space dimensions. We assemble a number of such results in this article, which is the second in a series of three articles [2, 3] on local refinement and multilevel preconditioners (the material forming this trilogy is based on the first author's Ph.D. dissertation [1]). This second article focuses on the WHB methods; the first article [3] developed some results for the BPX preconditioner.

The problem class we focus on here is linear second order partial differential equations (PDE) of the form:

$$-\nabla \cdot (p \; \nabla u) + q \; u = f, \quad u \in H_0^1(\Omega). \tag{1.1}$$

Here, $f \in L_2(\Omega)$, $p, q \in L_\infty(\Omega)$, $p : \Omega \to L(\mathbb{R}^d, \mathbb{R}^d)$, $q : \Omega \to \mathbb{R}$, where p is a symmetric positive definite matrix function, and where q is a nonnegative function. Let $\mathcal{T}_0$ be a shape regular and quasiuniform initial partition of Ω into a finite number of d simplices, and generate $\mathcal{T}_1, \mathcal{T}_2, \ldots$ by refining the initial partition using either red-green or red local refinement strategies in $d = 2$ or $d = 3$ spatial dimensions. Denote as $\mathcal{S}_j$ the simplicial linear C^0 finite element space corresponding to $\mathcal{T}_j$ equipped with zero boundary values. The set of nodal basis functions for $\mathcal{S}_j$ is denoted by $\Phi^{(j)} = \{\phi_i^{(j)}\}_{i=1}^{N_j}$ where $N_j = \dim \mathcal{S}_j$ is equal to the number of interior nodes in $\mathcal{T}_j$. Successively refined finite element spaces will form the following nested sequence:

$$\mathcal{S}_0 \subset \mathcal{S}_1 \subset \ldots \subset \mathcal{S}_j \subset \ldots \subset H_0^1(\Omega). \tag{1.2}$$

Although the mesh is nonconforming in the case of red refinement, $\mathcal{S}_j$ is used within the framework of conforming finite element methods for discretizing (1.1).

Let the bilinear form and the functional associated with the weak formulation of (1.1) be denoted as

$$a(u, v) = \int_\Omega p \; \nabla u \cdot \nabla v + q \; u \; v \; dx, \quad b(v) = \int_\Omega f \; v \; dx, \quad u, v \in H_0^1(\Omega).$$

We consider primarily the following Galerkin formulation: Find $u \in \mathcal{S}_j$, such that

$$a(u, v) = b(v), \quad \forall v \in \mathcal{S}_j. \tag{1.3}$$

The finite element approximation in $\mathcal{S}_j$ has the form $u^{(j)} = \sum_{i=1}^{N_j} u_i \phi_i^{(j)}$, where $u = (u_1, \ldots, u_{N_j})^T$ denotes the coefficients of $u^{(j)}$ with respect to $\Phi^{(j)}$. The resulting *discretization operator* $A^{(j)} = \{a(\phi_k^{(j)}, \phi_l^{(j)})\}_{k,l=1}^{N_j}$ determines the interaction of basis functions with respect to $a(\cdot, \cdot)$ and must be inverted numerically to determine the coefficients u from the linear system:

$$A^{(j)} u = F^{(j)}, \tag{1.4}$$

where $F^{(j)} = \{b(\phi_l^{(j)})\}_{l=1}^{N_j}$. Our task is to solve (1.4) with optimal (linear) complexity in both storage and computation, where the finite element spaces $\mathcal{S}_j$ are built on locally refined meshes. The condition number $\kappa_{\Phi^{(j)}}(A^{(j)})$ of $A^{(j)}$ with respect to the chosen basis $\Phi^{(j)}$ provides an upper bound on the number of iterations required by conjugate gradient-type methods to produce an approximate solution (satisfying a given fixed tolerance) to a linear system involving $A^{(j)}$. Therefore, it is desirable to have an analysis framework for bounding the condition number produced by a given basis, with the goal of finding bases which produce uniformly bounded condition numbers (or at least condition numbers with slow growth).

HB methods are particularly attractive in the local refinement setting because (by construction) each iteration has linear (optimal) computational and storage complexity. Unfortunately, the resulting preconditioner is not optimal due to condition number growth: in two dimensions the growth is slow, and the method is quite effective (nearly optimal), but in three dimensions the condition number grows much more rapidly with the number of unknowns. To address this instability, one can employ L_2-orthonormal wavelets in place of the hierarchical basis; such wavelets form a stable Riesz bases in H^1, thereby giving rise to an optimal preconditioner [13]. However, the complicated nature of traditional wavelet bases, in particular the non-local support of the basis functions and problematic treatment of boundary conditions, severely limits computational feasibility. WHB methods have been developed [23, 24] as an alternative, and they can be interpreted as a wavelet modification (or *stabilization*) of the hierarchical basis. These methods have been shown to optimally stabilize the condition number of the systems arising from hierarchical basis methods on quasiuniform meshes in both two and three space dimensions, and retain a comparable cost per iteration.

The framework developed in [23, 24] for the analysis of stabilizations of the hierarchical basis on quasiuniform meshes relies on establishing an optimal BPX preconditioner. In this article, we adopt the modern framework which exploits estimates related to depth of the hierarchy rather than the element size (i.e. 2^{-j} versus h). This framework enables the extension to local refinement setting. To use the extended framework, one again begins by establishing optimality of the BPX preconditioner, but now for the particular local refinement procedures of interest. One can find two such general optimality results for the BPX preconditioner on locally refined meshes in the literature. These are due to Dahmen and Kunoth [11] and Bornemann and Yserentant [7], both of which consider only the two-dimensional case. A third distinct set of results for the BPX preconditioner can be found in the companion article [3], which gives a comprehensive survey of the existing results, and also extends several of the existing BPX results to the three-dimensional local refinement setting. We use the approximation theory framework and optimal BPX results from [3] in this article to establish optimality results for WHB methods on locally refined meshes produced using two- and three-dimensional red-green and red refinement procedures. These local refinement procedures are fairly standard and can be easily implemented.

Outline of the paper. In §2, we review the relationship between condition numbers of matrices and stable Riesz bases. In §3, we outline a theoretical framework for constructing optimal multilevel preconditioners through decompositions of finite element spaces, giving necessary conditions on the decomposition operators for optimality. In §4, condition number bounds for the HB and WHB preconditioners are given by establishing explicit Riesz basis stability bounds, and we show that H^1-stability of the slice operators π_j is a necessary condition for obtaining a H^1-stable Riesz basis (or equivalently, an optimal preconditioner). In §5, we briefly describe implementable versions of red-green and red local refinement of two- and three-dimensional simplex meshes, and list a number of critical geometrical results for the resulting refined meshes that were established in [3]. In §6, we set up the main theoretical results in the paper, state the fundamental assumption for establishing basis stability and WHB preconditioner optimality, and establish the main results, namely the optimality of the WHB preconditioner in the 2D and 3D local refinement settings described in §5. The results in §6 rest completely on the BPX results from the companion article [3] and on Bernstein estimates, the latter of which rest on the geometrical results established in §5. In §7.1 the additive WHB method is analyzed, where optimal spectral equivalence is established for general PDE coefficients $p \in L_\infty(\Omega)$ for all the five local refinement procedures: 3D/2D red-green, 3D/2D red, and 2D red as in [7]. In §7.2, the multiplicative WHB method is considered, and optimality is shown for smooth PDE coefficients $p \in C^1(\Omega)$ for the 3D/2D red refinements in [3] and in [7]. A comparison of the two-dimensional red refinement procedures is given in §8. The theoretical obstacle for optimal multiplicative methods is the strengthened Cauchy-Schwarz inequality. However, in the absence of the strengthened Cauchy-Schwarz inequality or $p \in C^1(\Omega)$, we are able to show a nearly optimal spectral equivalence result by using the H^1-stability of W_j established in §4. Table 1.1 encapsulates the optimality results we establish in this article. A collection of experiments with the methods under consideration is presented in [2].

Table 1.1

Collection of optimality results proved in this article for the wavelet modified hierarchical basis methods. r, r-g, and opt. stand for red, red-green, and optimal respectively.

coeff.	method	3D r	2D r	2D r in [7]	3D r-g	2D r-g
$p \in C^1$	additive	opt.	opt.	opt.	opt.	opt.
	multiplicative	opt.	opt.	opt.	subopt.	subopt.
$p \in L_\infty$	additive	opt.	opt.	open	subopt.	subopt.
	multiplicative	subopt.	subopt.	open	subopt.	subopt.

Finally, as optimality of the WHB preconditioner implies H^1-stability of the W_j operator restricted to finite element spaces under the same class of local refinement algorithms, likewise a surprising implication of the optimality of the BPX preconditioner is H^1-stability of L_2-projection. This question has been actively studied in the finite element community due to its relationship to multilevel preconditioning. The existing theoretical results, mainly due to Carstensen [10] and Bramble-Pasciak-Steinbach [8] involve *a posteriori* verification of somewhat complicated mesh conditions after refinement has taken place. If such mesh conditions are not satisfied, one has to redefine the mesh. However, the stability result we obtained in §4.1 appears to be the first *a priori* H^1-stability result for L_2-projection on the finite element spaces produced.

2. Condition numbers and Riesz Bases. Let H be a separable Hilbert space with a nested sequence of finite dimensional subspaces,

$$H_0 \subset H_1 \subset \ldots \subset H_j \subset \ldots \subset H,$$

where $\dim(H_j) = N_j$. Consider a bounded bilinear form $a(\cdot,\cdot)$ defined on $H \times H$ satisfying the inf-sup condition. Let $u \in H_j$ and let $\Phi^{(j)} = \{\phi_i\}_{i=1}^{N_j}$ be a basis for H_j such that $u = \sum_{i=1}^{N_j} u_i \phi_i$, where $u = (u_1, \ldots, u_{N_j})^T$ denotes the coordinates of u with respect to $\Phi^{(j)}$. Let $A^{(j)} = \{a(\phi_k, \phi_l)\}_{k,l=1}^{N_j}$ denote the discretization operator with respect to $\Phi^{(j)}$. As remarked earlier, we are generally interested in the condition number of $A^{(j)}$ for different choices of bases, such as hierarchical-type bases.

A basis-dependent inner-product in the coefficient space will be used for the calculation of $\kappa_{\Phi^{(j)}}(A^{(j)})$, $\langle u, v\rangle_{\Phi^{(j)}} = \sum_{i=1}^{N_j} u_i v_i$, and the norm induced by $\langle \cdot, \cdot\rangle_{\Phi^{(j)}}$ will be denoted as $\|u\|^2_{\Phi^{(j)}} = \sum_{i=1}^{N_j} u_i^2$. Note that $\kappa_{\Phi^{(j)}}(A^{(j)})$ becomes uniformly bounded if $\Phi^{(j)}$ chosen to be an orthonormal basis with respect to the inner-product $(\cdot,\cdot)_H$ of H. However, it is not practical to assume the existence of an orthonormal basis which is also computationally feasible. In a separable Hilbert space H, the next best thing to an orthonormal basis, in this sense, is an H-stable Riesz basis.

Definition 2.1. *Let $\Phi = \{\phi_i\}_{i=1}^{\infty}$ be a basis for H, and $u = \sum_{i=1}^{\infty} c_i\phi_i$. If there exist two absolute constants σ_1 and σ_2 such that*

$$\sigma_1\|u\|^2_H \leq \sum_{i=1}^{\infty} c_i^2 \leq \sigma_2\|u\|^2_H, \quad \forall u \in H, \tag{2.1}$$

then Φ is called an H-stable Riesz basis.

The condition (2.1) for finite dimensional H_j can be written as

$$\sigma_1^{(j)} \leq \frac{\|u\|^2_{\Phi^{(j)}}}{\|u\|^2_{H_j}} \leq \sigma_2^{(j)}, \quad \forall u \in H_j. \tag{2.2}$$

The primary task becomes gaining some control over the ratio $\sigma_2^{(j)}/\sigma_1^{(j)}$.

Definition 2.2. *The family $\left\{\Phi^{(j)} \equiv \{\phi_i\}_{i=1}^{N_j}\right\}$ is a uniformly H_j-stable family of Riesz bases if there exists c independent of j such that: $\sigma_2^{(j)}/\sigma_1^{(j)} \leq c, \quad j \to \infty$.*

The case of primary interest is when $H_j = \mathcal{S}_j$. The discussion above results in the following theorem.

Theorem 2.3. *Let $\Phi^{(j)}$ be a basis of $\mathcal{S}_j$ satisfying (2.2). Then with c depending on the norm of the bilinear form and the stability constant from the inf-sup condition;*

$$\kappa_{\Phi^{(j)}}(A^{(j)}) \leq c\ \sigma_2^{(j)}/\sigma_1^{(j)}.$$

Note that $\sigma_2^{(j)}/\sigma_1^{(j)}$ is basis-dependent and our motivation is to find H^1-stable Riesz bases so that the condition number is uniformly bounded.

3. Multilevel preconditioning framework and the WHB preconditioner. The primary goal of this work is to describe an approximation theory framework for constructing and analyzing multilevel preconditioners, and then to use the framework to show that the wavelet modified hierarchical basis (WHB) preconditioner is optimal for several practical local refinement algorithms. Multilevel preconditioning exploits the underlying multilevel hierarchical structure. Let $\mathcal{N}_j^f$ denote the newly introduced (fine) nodes in a locally refined mesh, then the following decomposition at level j is naturally introduced:

$$\mathcal{N}_j = \mathcal{N}_{j-1} \cup \mathcal{N}_j^f. \tag{3.1}$$

The key point is to reflect the hierarchical ordering of nodes (3.1) in the corresponding nodal basis functions, thereby reaching a hierarchical splitting:

$$\mathcal{S}_j = \mathcal{S}_{j-1} \oplus \mathcal{S}_j^f, \tag{3.2}$$

where $\mathcal{S}_j^f$ is called a *slice space* (superscript f stands for *fine* and later c will stand for *coarse*). The two-level decomposition is central to HB methods [5]. In this process the slice space $\mathcal{S}_j^f$ is selected as a hierarchical complement of $\mathcal{S}_{j-1}$ in $\mathcal{S}_j$. Namely

$$\mathcal{S}_j^f = (\pi_j - \pi_{j-1})\mathcal{S}_j, \tag{3.3}$$

where $\pi_j : L_2 \rightarrow \mathcal{S}_j$ is a linear operator with the following three properties:

$$\pi_j \mid_{\mathcal{S}_j} = I, \tag{3.4}$$

$$\pi_j \pi_k = \pi_{\min\{j,k\}}, \tag{3.5}$$

$$\|(\pi_j - \pi_{j-1}) u^{(j)}\|_{L_2} \simeq \|u^{(j)}\|_{L_2}, \quad u^{(j)} \in (I_j - I_{j-1})\mathcal{S}_j, \tag{3.6}$$

where $I_j : L_2(\Omega) \rightarrow \mathcal{S}_j$ denotes the finite element interpolation operator. Applying the two-level decomposition (3.2) may not give a stable splitting of $\mathcal{S}_j$. This means that $A^{(j-1)}$ may not be well-conditioned. This difficulty can be overcome by repeating the above procedure so that $\mathcal{S}_J$ can be represented completely by slice spaces:

$$\mathcal{S} = \mathcal{S}_J = \mathcal{S}_0 \oplus \mathcal{S}_1^f \oplus \ldots \oplus \mathcal{S}_J^f. \tag{3.7}$$

Such a splitting will turn out not only to be stable, but as a consequence it will also have the advantage of producing well-conditioned fine-fine interaction operators $A_{22}^{(j)}$ as will be explained in §9.1. In light of (3.7), multilevel preconditioning can be interpreted as a *stable* splitting of $u \in \mathcal{S}_J$,

$$u = \sum_{j=0}^{J} (\pi_j - \pi_{j-1}) u. \tag{3.8}$$

The splitting (3.8) will then define a preconditioner $B^{(J)}$ with $\pi_{-1} = 0$:

$$(B^{(J)} u, v) \equiv \sum_{j=0}^{J} 2^{2j} ((\pi_j - \pi_{j-1}) u, (\pi_j - \pi_{j-1}) v), \quad u, v \in \mathcal{S}_J. \tag{3.9}$$

Let us assume that the inversion of $B^{(J)}$ is computationally feasible. If the following spectral equivalence can be established:

$$\lambda_{B^{(J)}} (B^{(J)} u, u) \leq (A^{(J)} u, u) \leq \Lambda_{B^{(J)}} (B^{(J)} u, u), \tag{3.10}$$

then the efficiency of the preconditioner will be determined by the ratio $\frac{\Lambda_{B^{(J)}}}{\lambda_{B^{(J)}}}$, since $\kappa({B^{(J)}}^{-1} A^{(J)}) \leq \frac{\Lambda_{B^{(J)}}}{\lambda_{B^{(J)}}}$. The preconditioner $B^{(J)}$ in (3.9) induces the so-called *preconditioner norm* as given below:

$$\|u\|_{B^{(J)}}^2 \equiv (B^{(J)} u, u) = \sum_{j=0}^{J} 2^{2j} \|(\pi_j - \pi_{j-1}) u\|_{L_2}^2. \tag{3.11}$$

Here, we should clarify that by *stable* splitting, we mean that the corresponding preconditioner will have favorable $\lambda_{B^{(J)}}$ and $\Lambda_{B^{(J)}}$, and in the best case, *optimal* bounds [18, 19].

Let $Q_j : L_2(\Omega) \rightarrow \mathcal{S}_j$ denote the L_2-projection. We are going to apply this framework to different examples by selecting π_j equal to I_j and Q_j, which will give rise to HB and BPX preconditioners, respectively. In local refinement, HB methods enjoy an optimal complexity of $O(N_j - N_{j-1})$ per iteration per level (resulting in $O(N_J)$ overall complexity per iteration) by only using degrees of freedom (DOF) corresponding to $\mathcal{S}_j^f$ by the virtue of (3.3). However, HB methods suffer from suboptimal iteration counts or equivalently suboptimal condition number. On the other hand, the BPX preconditioner enjoys an optimal condition number in the case of uniform refinement in 2D and 3D. In the companion article [3], we also showed that the optimal condition number extends to 2D/3D red-green and red refinement procedures. The BPX decomposition $\mathcal{S}_j = \mathcal{S}_{j-1} \oplus (Q_j - Q_{j-1})\mathcal{S}_j$ gives rise to basis functions which are not locally supported, but they decay rapidly outside a local support region. This allows for locally supported approximations, and in addition the WHB methods [23, 24, 25] can be viewed as an approximation of the wavelet basis stemming from the BPX decomposition [13]. A similar wavelet-like multilevel decomposition approach was taken in [22], where the orthogonal decomposition is formed by a discrete L_2-equivalent inner product. This approach utilizes the same BPX two-level decomposition [21, 22].

The WHB preconditioner introduced in [23, 24] is, in some sense, the best of both worlds. While the condition number of the HB preconditioner is stabilized by inserting Q_j in the definition of π_j, somehow employing the operators $I_j - I_{j-1}$ at the same time guarantees optimal computational and storage cost per iteration. The operators which will be seen to meet both goals at the same time are:

$$W_k = \prod_{j=k}^{J-1} I_j + Q_j^a(I_{j+1} - I_j), \tag{3.12}$$

with $W_J = I$. The exact L_2-projection Q_j is replaced by a computationally feasible approximation $Q_j^a : L_2 \to \mathcal{S}_j$. To control the approximation quality of Q_j^a, a small fixed tolerance γ is introduced:

$$\|(Q_j^a - Q_j)u\|_{L_2} \leq \gamma \|Q_j u\|_{L_2}, \quad \forall u \in L_2(\Omega). \tag{3.13}$$

In the limiting case $\gamma = 0$, W_k reduces to the exact L_2-projection on $\mathcal{S}_J$ by (3.4):

$$W_k = Q_k \;\; I_{k+1}Q_{k+1} \dots I_{J-1}Q_{J-1} \;\; I_J = Q_k Q_{k+1} \dots Q_{J-1} = Q_k.$$

The properties (3.4), (3.5), and (3.6) can be verified for W_k as follows:

• Property (3.4): Let $u^{(k)} \in \mathcal{S}_k$. Since $(I_{j+1} - I_j)u^{(k)} = 0$ and $I_j u^{(k)} = u^{(k)}$ for $k \leq j$, then $[I_j + Q_j^a(I_{j+1} - I_j)](u^{(k)}) = u^{(k)}$, verifying (3.4) for W_k. It also implies

$$W_k^2 = W_k. \tag{3.14}$$

• Property (3.5): Let $k \leq l$, then by (3.14)

$$W_k W_l = [(I_k + Q_k^a(I_{k+1} - I_k)) \dots (I_{l-1} + Q_{l-1}^a(I_l - I_{l-1})) \; W_l] W_l = W_k. \tag{3.15}$$

Since $W_k u \in \mathcal{S}_k$ and $\mathcal{S}_k \subset \mathcal{S}_l$, then by (3.4) we have

$$W_l(W_k u) = W_k u. \tag{3.16}$$

Finally, (3.5) then follows from (3.15) and (3.16).

• Property (3.6): This is an implication of Lemma 9.1.

The optimality of the WHB preconditioner in the locally refined cases is the main result of this paper (see Theorem 6.2). In particular, we establish the following norm equivalence:

$$\|u\|_{\text{WHB}}^2 \equiv \sum_{j=0}^{J} 2^{2j} \|(W_j - W_{j-1})u\|_{L_2}^2 \simeq \|u\|_{H^1}^2,$$

where W_j is as in (3.12), and where the underlying finite element spaces are built on fairly standard types of locally refined meshes. For an overview, we list the corresponding slice spaces for the preconditioners of interest:

$$\begin{array}{llcl} \text{HB:} & \mathcal{S}_j^f & = & (I_j - I_{j-1})\mathcal{S}_j, \\ \text{BPX:} & \mathcal{S}_j^f & = & (Q_j - Q_{j-1})\mathcal{S}_j, \\ \text{WHB:} & \mathcal{S}_j^f & = & (W_j - W_{j-1})\mathcal{S}_j = (I - Q_{j-1}^a)(I_j - I_{j-1})\mathcal{S}_j. \end{array}$$

4. H^1-stable Riesz bases and the WHB preconditioner. As the multilevel decomposition (3.7) suggests, one can view $\mathcal{S}_J$ as a span of multilevel hierarchical basis (MHB) functions. The MHB can be any computationally feasible basis and it is the nodal basis $\phi_i^{(j)}$ in our context. Modification to the nodal basis can be made by any linear operator π_j satisfying the properties (3.4), (3.5), and (3.6), in particular by the WHB operator W_j given in (3.12).

DEFINITION 4.1. *Let $\{\phi_i^{(j)}\}_{i=1}^{N_j}$ be the hierarchical basis for $\mathcal{S}_j$, $j = 0, \ldots, J$. Then the wavelet modified multilevel hierarchical basis (WMHB) for $\mathcal{S}_j$ is defined as follows:*

$$\Phi^{(J)} = \bigcup_{j=0}^{J} \left\{ (W_j - W_{j-1})\phi_i^{(j)} \right\}_{i=N_{j-1}+1}^{N_j}. \tag{4.1}$$

It can be shown (see Lemma 3.1 in [23]) that the WMHB (4.1) forms a basis for $\mathcal{S}_J$. With this fact at our disposal, let u be represented with respect to the WMHB:

$$u = \sum_{j=0}^{J} \sum_{i=N_{j-1}+1}^{N_j} c_i (W_j - W_{j-1})\phi_i^{(j)}. \tag{4.2}$$

Property (3.5) leads to:

$$W_k u = \sum_{j=0}^{k} \sum_{i=N_{j-1}+1}^{N_j} c_i (W_j - W_{j-1})\phi_i^{(j)}. \tag{4.3}$$

In order to establish Riesz stability, we will need a scaled version of the WMHB in (4.1) given as below:

$$\bar{\Phi}^{(J)} = \bigcup_{j=0}^{J} \left\{ 2^{j/2(d-2)} (W_j - W_{j-1})\bar{\phi}_i^{(j)} \right\}_{i=N_{j-1}+1}^{N_j}, \tag{4.4}$$

where $u = \sum_{j=0}^{J} \sum_{i=N_{j-1}+1}^{N_j} \bar{c}_i \bar{\phi}_i^{(j)} = \sum_{j=0}^{J} \sum_{i=N_{j-1}+1}^{N_j} c_i \phi_i^{(j)}$ and the following coefficient relationship holds:

$$\bar{c}_i = 2^{j/2(2-d)} c_i, \quad i = N_{j-1}+1, \ldots, N_j, \quad j = 0, \ldots, J. \tag{4.5}$$

The preconditioner norm $\| \cdot \|_{B^{(J)}}$ in (3.11) will then be equivalent to the coefficient norm $\| \cdot \|_{\bar{\Phi}^{(J)}}$. This norm equivalence can be expressed succinctly as follows:

LEMMA 4.2. *Let $u = \sum_{j=0}^{J} \sum_{i=N_{j-1}+1}^{N_j} \bar{c}_i \; 2^{j/2(d-2)} (\pi_j - \pi_{j-1}) \bar{\phi}_i^{(j)}$ and let π_j satisfy the properties (3.4), (3.5), and (3.6). Then*

$$\|u\|_{B^{(J)}}^2 \equiv \sum_{j=0}^{J} 2^{2j} \|(\pi_j - \pi_{j-1})u\|_{L_2}^2 \simeq \sum_{i=1}^{N_J} \bar{c}_i^2 \equiv \|u\|_{\bar{\Phi}^{(J)}}^2. \tag{4.6}$$

Proof. Using (4.3) and linearity of π_j respectively:

$$(\pi_j - \pi_{j-1})u = \sum_{i=N_{j-1}+1}^{N_j} c_i(\pi_j - \pi_{j-1})\phi_i^{(j)} = (\pi_j - \pi_{j-1}) \sum_{i=N_{j-1}+1}^{N_j} c_i \phi_i^{(j)}.$$

Note that $\sum_{i=N_{j-1}+1}^{N_j} c_i\phi_i^{(j)} \in (I_j - I_{j-1})\mathcal{S}_j$. Then by property (3.6)

$$\|(\pi_j - \pi_{j-1})u\|_{L_2}^2 \simeq \| \sum_{i=N_{j-1}+1}^{N_j} c_i\phi_i^{(j)}\|_{L_2}^2.$$

The mass matrix is equivalent to its diagonal due to shape regularity and compact support of basis functions. Moreover for $i = N_{j-1}+1, \ldots, N_j, \quad j = 0, \ldots, J$, the local refinements under consideration promise a quasiuniform support of $\phi_i^{(j)}$ (see (5.4)), hence $\|\phi_i^{(j)}\|_{L_2}^2 \simeq 2^{-jd}$. Putting these facts together, one gets:

$$\| \sum_{i=N_{j-1}+1}^{N_j} c_i\phi_i^{(j)}\|_{L_2}^2 \simeq \sum_{i=N_{j-1}+1}^{N_j} c_i^2\|\phi_i^{(j)}\|_{L_2}^2 \simeq \sum_{i=N_{j-1}+1}^{N_j} c_i^2\ 2^{-jd}.$$

Eventually by (4.5),

$$\sum_{j=0}^{J} 2^{2j}\|(\pi_j - \pi_{j-1})u\|_{L_2}^2 \simeq \sum_{j=0}^{J} 2^{j(2-d)} \sum_{i=N_{j-1}+1}^{N_j} c_i^2 = \sum_{i=1}^{N_J} \bar{c}_i^2.$$

□

There are two important connections here to H^1-stable Riesz bases. First, the equivalence (4.6) implies that constructing an optimal preconditioner is equivalent to forming an H^1-stable Riesz basis $\bar{\Phi}^{(J)}$. The involvement of π_j in both the splitting (3.8) and in the WMHB representation in (4.2) makes it the most crucial element in the stabilization. We then come to the central question: Which choice of π_j can make MHB an H^1-stable Riesz basis? The second connection to H^1-stable Riesz bases is the following theorem, which sets a guideline for picking π_j. It shows that H^1-stability of the π_j is actually a *necessary condition* for obtaining an optimal preconditioner.

THEOREM 4.3. *If $\pi_j : L_2 \to \mathcal{S}_j, \quad j = 0, \ldots, J$ give rise to an optimal preconditioner (or equivalently, if $\bar{\Phi}^{(J)}$ is an H^1-stable Riesz basis for $\mathcal{S}_J$), then for all $u \in \mathcal{S}_J$ there exists an absolute constant c such that*

$$\|\pi_j u\|_{H^1} \le c\ \|u\|_{H^1}, \quad \forall j \le J.$$

Proof. See Theorem 4 in [24]. □

The finite element interpolation operator I_j is not bounded in the H^1-norm, and the following explicit tight bounds are well-known [4, 6, 17, 28]:

$$\|I_j u\|_{H^1} \le c \left\{ \begin{array}{ll} (J-j+1)^{1/2}, & d = 2 \\ 2^{(J-j)/2}, & d = 3 \end{array} \right\} \|u\|_{H^1}.$$

In the light of Theorem 4.3, the basis in the HB method [6, 27] cannot form an

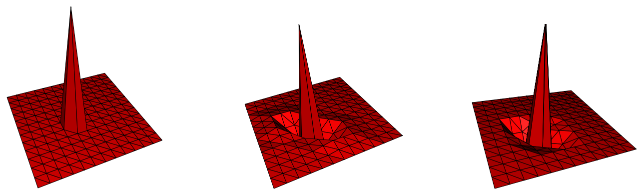

FIG. 4.1. *Left: Hierarchical basis function without modification. Wavelet modified hierarchical basis functions. Middle: One iteration of symmetric Gauss-Seidel approximation. Right: One iteration of Jacobi approximation.*

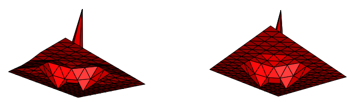

FIG. 4.2. *Lower view of middle and left basis functions in Figure 4.1.*

H^1-stable Riesz basis. For the performance analysis of the HB preconditioner, we choose the suitably scaled MHB as in (4.4) and (4.5). Then, by Lemma 4.2,

$$\|u\|_{\mathrm{HB}}^2 \equiv \sum_{j=0}^{J} 2^{2j} \|(I_j - I_{j-1})u\|_{L_2}^2 \simeq \sum_{j=1}^{N_J} \bar{c}_i^2 \equiv \|u\|_{\bar{\Phi}^{(J)}}^2 .$$

The suboptimal bounds for I_j manifest themselves as in the following widely known result [16, 18] about HB.

$$c_1 \left\{ \begin{array}{ll} J^{-2}, & d = 2 \\ 2^{-J}, & d = 3 \end{array} \right\} \|u\|_{\mathrm{HB}}^2 \leq \|u\|_{H^1}^2 \leq c_2 \|u\|_{\mathrm{HB}}^2 .$$

Therefore, the HB preconditioner is not optimal, and its performance severely deteriorates in dimension $d = 3$. Furthermore, Theorem 2.3 implies that the discretization operator $\bar{A}^{(J)} = \{a(\bar{\phi}_k^{(J)}, \bar{\phi}_l^{(J)})\}_{k,l=1}^{N_J}$ with respect to the scaled HB cannot be well-conditioned with the following tight bounds:

$$\kappa_{\bar{\Phi}^{(J)}}(\bar{A}^{(J)}) \leq c \left\{ \begin{array}{ll} J^2, & d = 2 \\ 2^J, & d = 3 \end{array} \right\} .$$

On the other hand, Theorem 6.2 indicates that the WMHB in (4.1) forms an H^1-stable Riesz basis (see Corollary 6.3). Hence, by Theorem 2.3, the discretization operator relative to the scaled WMHB in (4.4) is well-conditioned: $\kappa_{\bar{\Phi}^{(J)}}(\bar{A}^{(J)}) \leq c$. Riesz stability is attained through wavelet modifications. In particular, the modification is made by subtracting from each HB function $\phi_i^{(j)} \in \mathcal{S}_j^f$ its approximate L_2-projection $Q_{j-1}^a \phi_i^{(j)}$ onto the coarse level $j-1$. Such modifications are depicted in Figures 4.1 and 4.2. Note that modification with symmetric Gauss Seidel approximation gives rise to basis functions with larger supports than the ones modified with Jacobi approximation.

4.1. H^1-stable L_2-projection. We present a crucial consequence of Theorem 4.3.

COROLLARY 4.4. *L_2-projection, $Q_j|_{\mathcal{S}_j} : L_2 \to \mathcal{S}_j$, restricted to $\mathcal{S}_j$ is H^1-stable on 2D and 3D locally refined meshes by red-green and red refinement procedures.*

Proof. Optimality of the BPX preconditioner on the above locally refined meshes is established in the companion article [3]. Application of Theorem 4.3 with Q_j proves the result. Alternatively, the same result can be obtained through Theorem 4.3 applied to the WHB framework. Theorem 6.2 will establish the optimality of the WHB preconditioner for the local refinement procedures. Hence, the operator W_j restricted to $\mathcal{S}_j$ is H^1-stable. Since W_j is none other than Q_j in the limiting case, we can also conclude the H^1-stability of the L_2-projection. ☐

Our stability result appears to be the first *a priori* H^1-stability for the L_2-projection on these classes of locally refined meshes. H^1-stability of L_2-projection is guaranteed for the subset $\mathcal{S}_j$ of $L_2(\Omega)$, not for all of $L_2(\Omega)$. This question is currently undergoing intensive study in the finite element and approximation theory community. The existing theoretical results, mainly in [8, 10], involve *a posteriori* verification of somewhat complicated mesh conditions after refinement has taken place. If such mesh conditions are not satisfied, one has to redefine the mesh. The mesh conditions mentioned require that the simplex sizes do not change drastically between regions of refinement. In this context, quasiuniformity in the support of a basis function becomes crucial. This type of local quasiuniformity is usually called as *patchwise quasiuniformity*. Local quasiuniformity requires neighbor generation relations as in (5.1) and (5.2), neighbor size relations, and shape regularity of the mesh. It was shown in [1] that patchwise quasiuniformity holds also for 3D marked tetrahedron bisection [14] and for 2D newest vertex bisection [15, 20]. Then these are promising refinement procedures for which H^1-stability of the L_2-projection can be established.

5. Red-green and red refinements. We present only the highlights of the red-green and red refinement procedures; more detail, including a number of technical details concerning the refinement procedures themselves, can be found in the preceding article [3]. The two-dimensional case is quite standard, so we only describe the more complicated three-dimensional case here. The level of a simplex $\tau \in \mathcal{T}_j$ is defined as

$$L(\tau) = \min\{j : \tau \in \mathcal{T}_j\}.$$

Let us denote the support of basis functions corresponding to $\mathcal{N}_j^f$ as Ω_j^f. For our analysis, we will have a quasiuniform triangulation on Ω_j^f. One can analogously introduce a triangulation hierarchy

$$\mathcal{T}_j^f \equiv \{\tau \in \mathcal{T}_j : L(\tau) = j\} = \mathcal{T}_j|_{\Omega_j^f}.$$

Simplices in $\mathcal{T}_j^f$ are exposed to uniform refinement, hence $\mathcal{T}_j^f$ becomes a quasiuniform tetrahedralization.

Red refinement as a stand-alone procedure creates new DOF by pairwise quadrasection or octasection. The resulting hanging nodes are not closed, and therefore cannot be DOF (see the middle mesh in Figure 5.1 where a new DOF is represented by a small square). The initial triangulation $\mathcal{T}_0$ gives rise to nested, but possibly nonconforming triangulations; see the middle mesh in Figure 5.1. A function $u \in \mathcal{S}_j$ is determined by its values at DOF. Hanging nodes are always midpoints of edges connecting two DOF. The values at hanging nodes are computed by linear interpolation

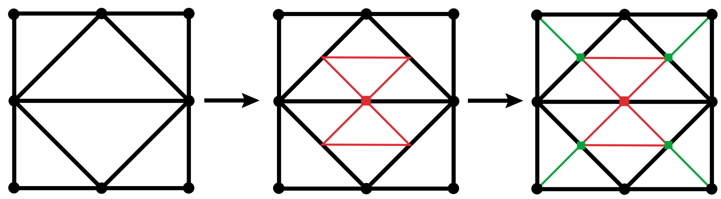

FIG. 5.1. *Left: Coarse DOF, $N_0 = 8$. Middle: a DOF created by red refinement, $N_1^{red} = 9$. Right: Green closure deployed, $N_1^{red-green} = 13$.*

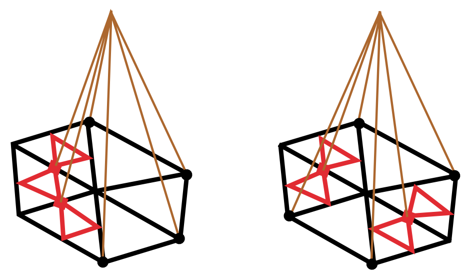

FIG. 5.2. *Basis functions on meshes created by two different red refinements. Left: Two DOF created on edge-adjacent simplices. Right: Two DOF created on non-edge-adjacent simplices.*

using the corresponding DOF at the ends of edges. Although the mesh is nonconforming, we have conforming, well-defined basis functions which satisfy the Lagrange property; see Figure 5.2.

A simplex in the red mesh can be expressed as a union of simplices in the corresponding red-green mesh. Then the red finite element space is a subspace of the corresponding red-green finite element space. Similarly, any simplex in $\mathcal{T}_j$ created by red refinement can be expressed as a union of simplices in the uniformly refined triangulations $\tilde{\mathcal{T}}_j$. (This property is no longer valid if red refinement is supplemented with the green refinement.) The simplex relationship gives rise to the most attractive property of red refinement: $\mathcal{S}_j$ is a true subspace of $\tilde{\mathcal{S}}_j$. This fact is quite convenient simply because the standard estimates such as inverse inequalities and Cauchy-Schwarz like estimates which naturally hold for $\tilde{\mathcal{S}}_j$ can be inherited for $\mathcal{S}_j$ without any additional effort. We will exploit this fact in proving the strengthened Cauchy-Schwarz inequality A.7.2 in the appendix.

The following generation bounds for neighbor simplices, established rigorously in [3], will be the foundation for the approximation theory estimates. Let τ and τ' be two d simplices in $\mathcal{T}_j$ sharing common d vertices. Then

$$\text{red-green refinement} : |L(\tau) - L(\tau')| \leq 1, \tag{5.1}$$

$$\text{red refinement} : |L(\tau) - L(\tau')| \leq 2. \tag{5.2}$$

The generation bounds (5.1) and (5.2) give rise to a L_2-stable Riesz basis in the following way [1, 3, 11]: Let the properly scaled nodal basis function be denoted as

$$\hat{\phi}_i^{(j)} = 2^{d/2L_{j,i}}\ \phi_i^{(j)}, \quad \hat{u}_i = 2^{-d/2L_{j,i}}\ u_i, \quad x_i \in \mathcal{N}_j,$$

where $L_{j,i} = \min\{L(\tau) : \tau \in \mathcal{T}_j,\ x_i \in \tau\}$. Then $\bigcup_{j=0}^{J} \{\hat{\phi}_i^{(j)}\}_{i=N_{j-1}+1}^{N_j}$ becomes a

L_2-stable Riesz basis [3]:

$$\| \sum_{x_i \in \mathcal{N}_j} \hat{u}_i \hat{\phi}_i^{(j)} \|_{L_2(\Omega)} \simeq \| \{\hat{u}_i\}_{x_i \in \mathcal{N}_j} \|_{l_2}. \tag{5.3}$$

Then (5.3) forms the sufficient condition to establish the Bernstein estimate:

$$\omega_2(u,t)_p \leq c\, (\min\{1, t2^J\})^{\beta} \|u\|_{L_p}, \quad u \in \mathcal{S}_J, \tag{5.4}$$

where $\omega_2(u,t)_p$ denotes second moduli of smoothness of u in L_p with step size t and $\beta > 1$. The constant c is independent of u and J. This crucial property helps us to prove Theorem 6.2.

6. Fundamental assumption, optimal preconditioner, and basis stability. As in the BPX splitting, the main ingredient in the WHB splitting is the L_2-projection. Hence, the stability of the BPX splitting is still important in the WHB splitting. The lower bound in the BPX norm equivalence is the *fundamental assumption* for both the additive and multiplicative WHB methods. Namely, there exists a constant σ independent of J satisfying:

ASSUMPTION 6.1.

$$\sum_{j=0}^{J} 2^{2j} \|(Q_j - Q_{j-1})u\|_{L_2}^2 \leq \sigma \|u\|_{H^1}^2, \quad \forall u \in \mathcal{S}_J.$$

A.6.1 was verified by the authors [1, 3] for 3D red-green and 2D/3D red refinement procedures. Dahmen and Kunoth [11] verified A.6.1 for 2D red-green refinement procedure. In addition, Bornemann and Yserentant [7] established A.6.1 for a different version of 2D red refinement procedure.

Before getting to the stability result we remark that the existing perturbation analysis of WHB is one of the primary insights in [23, 24]. Although not observed in [23, 24], the result does not require substantial modification for locally refined meshes. Let $e_j = (W_j - Q_j)u$ be the error, then the following holds.

LEMMA 6.1. *Let γ be as in (3.13). There exists an absolute c satisfying:*

$$\sum_{j=0}^{J} 2^{2j} \|e_j\|_{L_2}^2 \leq c\gamma^2 \sum_{j=0}^{J} 2^{2j} \|(Q_j - Q_{j-1})u\|_{L_2}^2, \quad \forall u \in \mathcal{S}_J. \tag{6.1}$$

Proof. See Lemma 5.1 and page 119 in [23] or Lemma 1 in [24]. ☐

We arrive now at the primary result, which indicates that the WHB preconditioner is optimal on the class of locally refined meshes under consideration.

THEOREM 6.2. *If there exists sufficiently small γ_0 such that (3.13) is satisfied for $\gamma \in [0, \gamma_0)$, then*

$$\|u\|_{\text{WHB}}^2 \equiv \sum_{j=0}^{J} 2^{2j} \|(W_j - W_{j-1})u\|_{L_2}^2 \simeq \|u\|_{H^1}^2, \quad u \in \mathcal{S}_J. \tag{6.2}$$

Proof. Observe that

$$\begin{aligned}(W_j - W_{j-1})u &= (W_j - Q_j)u - (W_{j-1} - Q_{j-1})u + (Q_j - Q_{j-1})u \\ &= e_j - e_{j-1} + (Q_j - Q_{j-1})u.\end{aligned} \tag{6.3}$$

This gives

$$\sum_{j=0}^{J} 2^{2j}\|(W_j - W_{j-1})u\|_{L_2}^2 \le c\sum_{j=0}^{J} 2^{2j}\|(Q_j - Q_{j-1})u\|_{L_2}^2 + c\sum_{j=0}^{J} 2^{2j}\|e_j\|_{L_2}^2$$
$$\le c(1+\gamma^2)\sum_{j=0}^{J} 2^{2j}\|(Q_j - Q_{j-1})u\|_{L_2}^2 \quad \text{(using (6.1))}$$
$$\le c\|u\|_{H^1}^2 \quad \text{(using A.6.1)}.$$

Let us now proceed with the upper bound. The Bernstein estimate (5.4) holds for $\mathcal{S}_j$ [1, 3, 11] for all the five local refinement procedures. Hence we are going to utilize an inequality involving the Besov norm $\|\cdot\|_{B^1_{2,2}}$ which naturally fits our framework when the moduli of smoothness is considered in (5.4). The following important inequality holds, provided that (5.4) holds (see page 39 in [19]):

$$\|u\|_{B^1_{2,2}}^2 \le c\sum_{j=0}^{J} 2^{2j}\|u^{(j)}\|_{L_2}^2, \tag{6.4}$$

for any decomposition such that $u = \sum_{j=0}^{J} u^{(j)}, \quad u^{(j)} \in \mathcal{S}_j$, in particular for $u^{(j)} = (W_j - W_{j-1})u$. Then the upper bound holds due to $H^1(\Omega) \cong B^1_{2,2}(\Omega)$. ▯

Remark 6.1. *The following equivalence is used for the upper bound in the proof of Theorem 6.2 on uniformly refined meshes (cf. Lemma 4 in [24]).*

$$c_1\|u\|_{H^1}^2 \le \inf_{u=\sum_{j=0}^{J} u^{(j)},\ u^{(j)}\in\mathcal{S}_j} \sum_{j=0}^{J} 2^{2j}\|u^{(j)}\|_{L_2}^2 \le c_1\|u\|_{H^1}^2.$$

Let us emphasize that the left hand side holds in the presence of the Bernstein estimate (5.4), and the right hand side holds in the simultaneous presence of Bernstein and Jackson estimates. However, the Jackson estimate cannot hold under local refinement procedures (cf. counter example in section 8 in [3]). That is why we can utilize only the left hand side of the above equivalence as in (6.4).

The WHB preconditioner optimality will be connected to Riesz basis and the scaled WMHB will now be a H^1-stable Riesz basis by Lemma 4.2 and Theorem 6.2.

Corollary 6.3. *Let u be represented with respect to $\bar{\Phi}^{(J)}$ in (4.4). If there exists $\gamma \in [0, \gamma_0)$ such that (3.13) holds, then $\bar{\Phi}^{(J)}$ forms an H^1-stable Riesz basis:*

$$\|u\|_{\bar{\Phi}^{(J)}}^2 \equiv \sum_{i=1}^{N_J} \bar{c}_i^2 \simeq \|u\|_{H^1}^2.$$

7. Optimality framework. The linear algebra setting of HB methods [6] has a corresponding operator setting. Namely, the discretization operator $A^{(j)} : \mathcal{S}_j \to \mathcal{S}_j$ and its restriction onto $\mathcal{S}_j^f$, fine discretization operator $A_{22}^{(j)} : \mathcal{S}_j^f \to \mathcal{S}_j^f$, are defined respectively as follows:

$$(A^{(j)}\varphi, \psi) = a(\varphi, \psi), \quad \forall \varphi, \psi \in \mathcal{S}_j, \quad (A_{22}^{(j)}\psi^f, \varphi^f) = a(\psi^f, \varphi^f), \quad \forall \varphi^f, \psi^f \in \mathcal{S}_j^f.$$

The communication operators $A_{12}^{(j)} : \mathcal{S}_j^f \to \mathcal{S}_{j-1}, \quad A_{21}^{(j)} : \mathcal{S}_{j-1} \to \mathcal{S}_j^f$ are given by:

$$(A_{12}^{(j)}\varphi^f, \psi^c) = (\varphi^f, A_{21}^{(j)}\psi^c) = a(\varphi^f, \psi^c), \quad \forall \psi^c \in \mathcal{S}_{j-1}, \quad \varphi^f \in \mathcal{S}_j^f.$$

Since the decomposition $\mathcal{S}_j = \mathcal{S}_{j-1} \oplus \mathcal{S}_j^f$ in (3.2) is direct, $A^{(j)}$ can be represented by a two-by-two block form:

$$A^{(j)} = \begin{bmatrix} A^{(j-1)} & A_{12}^{(j)} \\ A_{21}^{(j)} & A_{22}^{(j)} \end{bmatrix} \begin{matrix} \} & \mathcal{S}_{j-1} \\ \} & \mathcal{S}_j^f \end{matrix} , \tag{7.1}$$

where $A^{(j-1)}$, $A_{12}^{(j)}$, $A_{21}^{(j)}$, and $A_{22}^{(j)}$ correspond to coarse-coarse, coarse-fine, fine-coarse, and fine-fine interactions respectively. In any HB method smoothing is performed on the fine discretization operator $A_{22}^{(j)}$. Hence, existence of approximations $B_{22}^{(j)}$, SPD in $\mathcal{S}_j^f$, to the operators $A_{22}^{(j)}$, $j = 1, \ldots, J$, becomes the second assumption pertaining to the preconditioners. The reader can find the verification of this assumption in §9.1.

ASSUMPTION 7.1.

$$(A_{22}^{(j)} u^f, u^f) \leq (B_{22}^{(j)} u^f, u^f) \leq (1 + b_1)(A_{22}^{(j)} u^f, u^f), \quad \forall u^f \in \mathcal{S}_j^f.$$

Next, optimality proof of the additive and multiplicative WHB methods will be given. Optimality will be shown in the form of the following spectral equivalence.

$$(A^{(J)}u, u) \leq (B^{(J)}u, u) \leq C_{opt}\ (A^{(J)}u, u), \quad \forall u \in \mathcal{S}_J. \tag{7.2}$$

7.1. Optimality of the additive WHB methods. The first optimality result for WHB methods will be for the additive version.

DEFINITION 7.1. *The additive WHB method $D^{(j)}$ is defined for $j = 1, \ldots, J$ as*

$$D^{(j)} \equiv \begin{bmatrix} D^{(j-1)} & 0 \\ 0 & B_{22}^{(j)} \end{bmatrix},$$

with $D^{(0)} = A^{(0)}$. Then,

$$(D^{(J)}u, u) = (A^{(0)}W_0 u, W_0 u) + \sum_{j=1}^{J} (B_{22}^{(j)}(W_j - W_{j-1})u, (W_j - W_{j-1})u),$$

where $u = \sum_{j=0}^{J}(W_j - W_{j-1})u$ as in (3.7) and (3.8). Now, we have all the required estimates at our disposal to establish the optimality of the additive WHB method for 2D/3D red-green and 2D/3D red refinement procedures for $p \in L_\infty(\Omega)$. We would like to emphasize that our framework supports any spatial dimension $d \geq 1$, provided that the necessary geometrical abstractions are in place. Additionally, optimality of the additive WHB method holds for a different version of 2D red refinement procedure introduced in [7] with $p \in C^1(\Omega)$. The optimality of the additive WHB method for all the local refinement procedures discussed is as follows.

THEOREM 7.2. *If A.6.1 holds and if there exists sufficiently small γ_0 such that (3.13) is satisfied for $\gamma \in (0, \gamma_0)$, then $A^{(J)}$ is spectrally equivalent to $D^{(J)}$ with $C_{opt} = c$ in (7.2).*

Proof. By A.7.1, $B_{22}^{(j)}$ is spectrally equivalent to $A_{22}^{(j)}$. Since $A_{22}^{(j)}$ is a well-conditioned matrix, using (9.4) it is spectrally equivalent to $2^{2j}I$. Then, $(D^{(J)}u, u) \simeq \sum_{j=0}^{J} 2^{2j} \|(W_j - W_{j-1})u\|_{L_2}^2$. The result follows from Theorem 6.2. ▫

7.2. Optimality of the multiplicative WHB methods. The standard assumption for multiplicative Schwarz methods is a fundamental inequality in multilevel finite element theory. It is known as the strengthened Cauchy-Schwarz inequality [7, 26, 28]. Bornemann and Yserentant [7] established this inequality for a variant of 2D red refinement procedure with $p \in C^1(\Omega)$. We extended their result to 3D red refinement. Highlights of the proof are presented in the appendix (see §9.2).

ASSUMPTION 7.2. *For $\delta \in (0,1)$ and $i = 1,\ldots,J$:*

$$|a(u^{(i)}, u^{(j)})|^2 \leq \sigma\delta^{2(j-i)} 2^{2j} a(u^{(i)}, u^{(i)}) \|u^{(j)}\|^2_{L_2}, \quad \forall u^{(i)} \in \mathcal{S}_i,\ u^{(j)} \in \mathcal{S}_j,\ j \geq i.$$

The motivation behind the multiplicative WHB method is the standard block-Cholesky factorization.

DEFINITION 7.3. *The multiplicative WHB method $B^{(j)}$ is defined as*

$$B^{(j)} \equiv \begin{bmatrix} B^{(j-1)} & A^{(j)}_{12} \\ 0 & B^{(j)}_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ B^{(j)^{-1}}_{22} A^{(j)}_{21} & I \end{bmatrix} = \begin{bmatrix} B^{(j-1)} + A^{(j)}_{12} B^{(j)^{-1}}_{22} A^{(j)}_{21} & A^{(j)}_{12} \\ A^{(j)}_{21} & B^{(j)}_{22} \end{bmatrix}.$$

Using the two-by-two block definition of $A^{(j)}$ as in (7.1), we define the error operator $E^{(j)}$ as

$$E^{(j)} \equiv B^{(j)} - A^{(j)} = \begin{bmatrix} B^{(j-1)} - A^{(j-1)} + A^{(j)}_{12} B^{(j)^{-1}}_{22} A^{(j)}_{21} & 0 \\ 0 & B^{(j)}_{22} - A^{(j)}_{22} \end{bmatrix}.$$

To realize the action of the error operator, we decompose $u \in \mathcal{S}_j$ as $u = u^c + u^f$, where $u^c \in \mathcal{S}_{j-1},\ u^f \in \mathcal{S}^f_j$. The action of $E^{(j)}$ then can be characterized as follows:

$$(E^{(j)}u, u) = ((B^{(j)}_{22} - A^{(j)}_{22})u^f, u^f) + (E^{(j-1)}u^c, u^c) + (B^{(j)^{-1}}_{22} A^{(j)}_{21} u^c, A^{(j)}_{21} u^c).$$

Our intention is to formalize the spectral equivalence of $A^{(j)}$ and $B^{(j)}$ in terms of $E^{(j)}$. Next, we verify standard requirements and characterize $E^{(j)}$ by utilizing the direct decomposition (3.2). By using the fact that $B^{(j)}_{22}$ is SPD in $\mathcal{S}^f_j$, one can see that the operator $E^{(j)}$ is positive semidefinite. In general, $u^{(j)} \in \mathcal{S}_j$ has the decomposition

$$u^{(j)} = u^{(j-1)} + u^{(j)^f}, \quad u^{(j-1)} \in \mathcal{S}_{j-1},\ u^{(j)^f} \in \mathcal{S}^f_j. \tag{7.3}$$

Then using A.7.1 we get;

$$\begin{aligned}(E^{(j)}u^{(j)}, u^{(j)}) - (E^{(j-1)}u^{(j-1)}, u^{(j-1)}) &\leq b_1(A^{(j)}_{22} u^{(j)^f}, u^{(j)^f}) \\ &\quad + (B^{(j)^{-1}}_{22} A^{(j)}_{21} u^{(j-1)}, A^{(j)}_{21} u^{(j-1)}).\end{aligned}$$

Summing over j, with $u = u^{(J)}$

$$(E^{(J)}u, u) \leq b_1 \sum_{j=1}^{J} (A^{(j)}_{22} u^{(j)^f}, u^{(j)^f}) + \sum_{j=1}^{J} (B^{(j)^{-1}}_{22} A^{(j)}_{21} u^{(j-1)}, A^{(j)}_{21} u^{(j-1)}). \tag{7.4}$$

In order to relate the sums appearing in (7.4) to $(A^{(J)}u, u)$, we will employ inequalities (7.5) and (7.6) respectively.

$$\sum_{j=1}^{J}(A_{22}^{(j)}u^{(j)^f}, u^{(j)^f}) \le \rho_1(A^{(J)}u, u), \tag{7.5}$$

$$\sum_{j=1}^{J}(B_{22}^{(j)^{-1}}A_{21}^{(j)}u^{(j-1)}, A_{21}^{(j)}u^{(j-1)}) \le \rho_2(A^{(J)}u, u). \tag{7.6}$$

One arrives at the spectral equivalence of $A^{(J)}$ and $B^{(J)}$ operators after having all the three assumptions in place; A.7.1, (7.5), and (7.6).

THEOREM 7.4. *If A.7.1, inequalities (7.5) and (7.6) hold true, then (7.2) holds with* $C_{opt} = b_1\rho_1 + \rho_2$.

Proof. The first inequality is attained by positive semidefiniteness of $E^{(j)}$. The second one follows from inequalities (7.5) and (7.6). ▯

In order to establish the spectral equivalence (7.2), we rely on the inequalities (7.5) and (7.6). From this point on, we concentrate on verifying these inequalities for the local refinement procedures under consideration. Moreover, the generic decomposition (7.3) will be replaced by the decomposition of interest. Namely, we establish the optimality of the multiplicative WHB method for the following decomposition:

$$u^{(j)} = W_{j-1}u^{(j)} + (W_j - W_{j-1})u^{(j)} \equiv u^{(j-1)} + u^{(j)^f}, \quad u^{(j-1)} \in \mathcal{S}_{j-1},\ u^{(j)^f} \in \mathcal{S}_j^f.$$

LEMMA 7.5. *If A.6.1 holds and γ is sufficiently small in (3.13) then (7.5) holds for some constant ρ_1.*

Proof. Using (6.3), we get:

$$\|u^{(j)^f}\|_{L_2} \le \|(Q_j - Q_{j-1})u\|_{L_2} + \|e_j\|_{L_2} + \|e_{j-1}\|_{L_2}. \tag{7.7}$$

$$\begin{aligned}
\sum_{j=1}^{J}(A_{22}^{(j)}u^{(j)^f}, u^{(j)^f}) &\le c\sum_{j=1}^{J}2^{2j}\|u^{(j)^f}\|_{L_2}^2 \quad \text{(using inverse inequality for } \mathcal{S}_j^f\text{)}\\
&\le c\sum_{j=1}^{J}2^{2j}\|(Q_j - Q_{j-1})u\|_{L_2}^2 + c\sum_{j=1}^{J}2^{2j}\|e_j\|_{L_2}^2 \quad \text{(using (7.7))}\\
&\le c(A^{(J)}u, u) \quad \text{(using (6.1) and A.6.1)}
\end{aligned}$$

For the quasiuniform setting, see Lemma 5.2 in [23]. ▯

Let us verify A.7.1. (9.4) indicates that $A_{22}^{(j)}$ is well-conditioned. Thus, one may choose a diagonal preconditioner $B_{22}^{(j)} = \alpha 2^{2j} I$ for the matrix $A_{22}^{(j)}$. Here α is a parameter which should be adjusted so that A.7.1 is satisfied for some b_1. With the above selection of $B_{22}^{(j)}$, we get

$$\sum_{j=1}^{J}(B_{22}^{(j)^{-1}}A_{21}^{(j)}u^{(j-1)}, A_{21}^{(j)}u^{(j-1)}) \le c\sum_{j=1}^{J}2^{-2j}\|A_{21}^{(j)}u^{(j-1)}\|_{L_2}^2. \tag{7.8}$$

The remaining link to reach to (7.6) will be provided by the following.

LEMMA 7.6. *If A.6.1 and A.7.2 hold, γ is sufficiently small in (3.13), then (7.6) holds for some constant ρ_2.*

Proof. Observe that the following estimate holds:

$$\|A_{21}^{(j)} u^{(j-1)}\|_{L_2} \le \|A^{(j)} u^{(j-1)}\|_{L_2}. \tag{7.9}$$

Now, using (7.8) and (7.9), $u^{(j-1)} = e_{j-1} + Q_{j-1}u$, and the inverse inequality for $\mathcal{S}_j \subset \tilde{\mathcal{S}}_j$ respectively.

$$\begin{aligned}\sum_{j=1}^{J} (B_{22}^{(j)^{-1}} A_{21}^{(j)} u^{(j-1)}, A_{21}^{(j)} u^{(j-1)}) &\le c \sum_{j=1}^{J} 2^{-2j} \|A^{(j)} u^{(j-1)}\|_{L_2} \\ &\le c \sum_{j=1}^{J} 2^{-2j} \left(\|A^{(j)} e_{j-1}\|_{L_2}^2 + \|A^{(j)} Q_{j-1} u\|_{L_2}^2 \right) \\ &\le c \sum_{j=1}^{J} 2^{-2(j-1)} \|e_{j-1}\|_{L_2}^2 + c \sum_{j=1}^{J} 2^{-2j} \|A^{(j)} Q_{j-1} u\|_{L_2}^2.\end{aligned}$$

The result follows by applying (6.1) and A.6.1 to the first sum in the above estimate. The second sum requires A.7.2 and we apply the estimate in Lemma 4.2 in [23]. For the quasiuniform setting, see Lemma 5.3 in [23]. ▯

Finally, the optimality result follows:

THEOREM 7.7. *If A.6.1 and A.7.2 hold, and γ is sufficiently small in (3.13) then (7.2) holds with $C_{opt} = c$, where c depends only on b_1 from A.7.1, δ from A.7.2, and σ from A.6.1, A.7.2.*

Proof. Lemma 7.5 and Lemma 7.6 establish the inequalities (7.5) and (7.6), respectively. Then the optimality statement follows from Theorem 7.4. ▯

8. Comparison of red refinements and suboptimal estimates. We have seen in §7.1 that the optimality of the additive WHB method is established for each of the four different local refinement procedures examined in §5, namely 2D and 3D red-green, as well as 2D and 3D red refinement procedures, for $p \in L_\infty(\Omega)$ and in general, extension of this class of refinement procedures to any spatial dimension $d \ge 1$. In addition, the optimality holds for the 2D red refinement introduced by Bornemann and Yserentant [7] with $p \in C^1(\Omega)$. However, for the optimality of the multiplicative WHB method the main theoretical challenge is to establish A.7.2. For this reason, we concentrate on proving optimality for the following three red refinement procedures; 2D and 3D ones as in §5, and the 2D one as in [7].

Let us elaborate on the two different 2D red refinement procedures. The one in [7] enforces the difference of levels of two simplices to be *at most 1* if they have at least one common node. This brings a *patchwise uniform* refinement flavor and is closer to uniform refinement than the type of red refinement in §5. There is an advantage of this type of refinement: All the subsimplices of a subdivided simplex can be marked for further refinement. In our refinement, this holds only for a subset of the subsimplices. On the other hand, one can introduce DOF inside a given patch without uniformly refining the whole patch. This flexible behavior is exhibited in Figure 5.2. Our 2D red refinement guarantees the BPX optimality for $p \in L_\infty(\Omega)$. In addition, this framework supports an easy extension to any spatial dimension $d \ge 1$. The BPX optimality presented in [7] is restricted to $p \in C^1(\Omega)$ with $d = 2$.

In §5, we mentioned that any red refinement procedure is attractive because $\mathcal{S}_j$ is a subspace of $\tilde{\mathcal{S}}_j$. Aside from this fact, 2D red refinement provides the right framework for the proof of the strengthened Cauchy-Schwarz inequality which forms the challenging assumption A.7.2 (see Lemma 9.4). In particular, a boundary strip S of the triangle τ is utilized to allow the decomposition

$$w = w_{bdry} + w_{intr}, \tag{8.1}$$

where w_{bdry} live on S and w_{intr} live in the interior (complement of S). One has to make sure that S is nonempty to utilize (8.1). The resulting strip is contained by the strip $\tilde{S}$ arising in the uniformly refined case (i.e. $S \subset \tilde{S}$). The proof technique aims to obtain a ratio with $\delta \in (0,1)$ such that $\frac{\text{area}(S)}{\text{area}(\tau)} \leq \frac{\text{area}(\tilde{S})}{\text{area}(\tau)} \leq c\ \delta^{2(j-i)}$. This subtle property cannot be satisfied by red-green refinement. An other difficulty arises in the proof of A.7.2 when $p \notin C^1(\Omega)$ (see §9.2). Without assuming A.7.2, $B^{(J)}$ is suboptimally spectral equivalent to $A^{(J)}$ as in Theorem 8.2. One can derive the following suboptimal estimate.

LEMMA 8.1. *If A.6.1 holds, then there exists c such that*

$$\sum_{j=1}^{J} 2^{-2j} \|A_{21}^{(j)} u^{(j-1)}\|_{L_2}^2 \leq c\ J(A^{(J)}u, u), \quad \forall u \in \mathcal{S}_J.$$

Proof.

$$\begin{aligned}
2^{-2j}\|A_{21}^{(j)}u^{(j-1)}\|_{L_2}^2 &\leq 2^{-2j}\|A^{(j)}u^{(j-1)}\|_{L_2}^2 \quad \text{(using (7.9))}\\
&= 2^{-2j}a(A^{(j)}u^{(j-1)}, u^{(j-1)})\\
&\simeq a(u^{(j-1)}, u^{(j-1)}) \quad \text{(largest eigenvalues of } A^{(j)} \sim 2^{2j})\\
&\simeq \|u^{(j-1)}\|_{H^1}^2 \equiv \|W_{j-1}u\|_{H^1}^2\\
&\leq c\|u\|_{H^1}^2
\end{aligned}$$

Optimality of the WHB preconditioner (i.e. decomposition generated by using W_j) is guaranteed by Theorem 6.2. Hence, one obtains the last inequality by the H^1-stability of W_{j-1} provided by Theorem 4.3. This leads to suboptimal estimate. For uniform refinement setting, see Lemma 4.4 in [23]. ∎

For multiplicative methods, in the absence of A.7.2 one uses the H^1-stability of the linear operator employed. So, H^1-stability of W_j plays a crucial role in Lemma 8.1. This explains why we have dedicated §4.1 for stability results in H^1. Finally, we report the suboptimal norm equivalence results.

THEOREM 8.2. *If A.6.1 holds, then (7.2) holds with $C_{opt} = c(1 + J)$, where c depends only on b_1 from A.7.1, δ from A.7.2, and σ from A.6.1, and the H^1-norm of the linear operator W_j for $j = 0, \ldots, J$.*

Proof. Lemma 7.5 implies inequality (7.5). Lemma 8.1 establishes the suboptimal inequality (7.6). Then the suboptimal spectral equivalence follows from Theorem 7.4. ∎

Griebel and Oswald [12] gave an improved suboptimal result for quasiuniform settings where (7.2) holds with $C_{opt} = c(1 + \log_2(1 + J))$.

9. Appendix.

9.1. Well-conditioned $A_{22}^{(j)}$.

The lemma below is essential to extend the existing results for quasiuniform meshes (cf. Lemma 6.1 in [23] or Lemma 2 in [24]) to the locally refined ones. $\mathcal{S}_j^{(f)} = (I_j - I_{j-1})\mathcal{S}_j$ denotes the HB slice space.

LEMMA 9.1. *Let $\mathcal{T}_j$ be constructed by the local refinements under consideration. Let $\mathcal{S}_j^f = (I - \pi_{j-1})\mathcal{S}_j^{(f)}$ be the modified hierarchical subspace where π_{j-1} is any L_2-bounded operator. Then, there are constants c_1 and c_2 independent of j such that*

$$c_1 \|\phi^f\|_X^2 \leq \|\psi^f\|_X^2 \leq c_2 \|\phi^f\|_X^2, \quad X = H^1, L_2, \tag{9.1}$$

holds for any $\psi^f = (I - \pi_{j-1})\phi^f \in \mathcal{S}_j^f$ with $\phi^f \in \mathcal{S}_j^{(f)}$.

Proof. The Cauchy-Schwarz like inequality [5] is central to the proof: There exists $\delta \in (0,1)$ independent of the mesh size or level j such that

$$(1-\delta^2)(\nabla\phi^f, \nabla\phi^f) \leq (\nabla(\phi^c + \phi^f), \nabla(\phi^c + \phi^f)), \quad \forall \phi^c \in \mathcal{S}_{j-1}, \phi^f \in \mathcal{S}_j^{(f)}. \tag{9.2}$$

$$(1-\delta^2)\|\phi^f\|_{L_2}^2 \leq c|\phi^c + \phi^f|_{H^1}^2 \quad \text{(by Poincare inequality and (9.2))}. \tag{9.3}$$

Combining (9.2) and (9.3): $(1-\delta^2)\|\phi^f\|_{H^1}^2 \leq \|\phi^c + \phi^f\|_{H^1}^2$. Choosing $\phi^c = -\pi_{j-1}\phi^f$, we get the lower bound: $(1-\delta^2)\|\phi^f\|_{H^1}^2 \leq \|\psi^f\|_{H^1}^2$. To derive the upper bound:

- Red-green refinement: The inverse inequality holds for $\mathcal{S}_j^f$ because of the quasiuniformity of $\mathcal{T}_j^f$. The right scaling is obtained by father-son size relation.
- Red refinement: By $\mathcal{S}_j^f \subset \mathcal{S}_j \subset \tilde{\mathcal{S}}_j$, the local inverse inequality (9.5) holds.

Using the inverse inequalities and L_2-boundedness of π_{j-1}, one gets

$$\|\psi^f\|_{H^1}^2 \leq c_0 2^{2j} \|\psi^f\|_{L_2}^2 \leq c_0 2^{2j} \left(1 + \|\pi_{j-1}\|_{L_2}\right)^2 \|\phi^f\|_{L_2}^2 \leq c 2^{2j} \|\phi^f\|_{L_2}^2.$$

The slice space $\mathcal{S}_j^{(f)}$ is oscillatory. Then there exists c such that $\|\phi^f\|_{L_2}^2 \leq c 2^{-2j} \|\phi^f\|_{H^1}^2$. Hence, $\|\psi^f\|_{H^1}^2 \leq c\|\phi^f\|_{H^1}^2$. The case for $X = L_2$ can be established similarly. ∎

Using the above tools, one can establish that $A_{22}^{(j)}$ is well-conditioned. Namely,

$$c_1 2^{2j} \leq \lambda_{j,\min}^f \leq \lambda_{j,\max}^f \leq c_2 2^{2j}, \tag{9.4}$$

where $\lambda_{j,\min}^f$ and $\lambda_{j,\max}^f$ are the smallest and largest eigenvalues of $A_{22}^{(j)}$, and c_1 are and c_2 both independent of j. For details see Lemma 4.3 in [23] or Lemma 3 in [24].

9.2. The strengthened Cauchy-Schwarz inequality. A.7.2 will be verified for uniform refinement. Following the exposition in [7], we extend the results to 3 spatial dimensions. We report some necessary technical lemmas and proof highlights.

LEMMA 9.2. *Let $u \in \tilde{\mathcal{S}}_j$. The inverse inequality holds for $\tau \in \tilde{\mathcal{T}}_j$:*

$$|u|_{H^1(\tau)}^2 \leq c_0 2^{2j} \|u\|_{L_2(\tau)}^2, \tag{9.5}$$

where c_0 depends only on the shape regularity of $\mathcal{T}_0$.

As mentioned before, a boundary strip will be employed to prove A.7.2. This requires a cut-off operation of the functions $u \in \mathcal{S}_j$. The next lemma quantifies the L_2-norm of u under this operation for a general setting where $d \geq 2$.

LEMMA 9.3. *Let τ be a d simplex which is a subset of a simplex in $\mathcal{T}_0$. Let $\bar{u}$ be a linear function taking the same values as u at most $d-1$ vertices of τ and the value 0 at the remaining vertices of τ. Then the following sharp bound holds:*

$$\|\bar{u}\|_{L_2(\tau)}^2 \leq \frac{d+1}{2} \|u\|_{L_2(\tau)}^2. \tag{9.6}$$

Proof. Define $F(x_1, \ldots, x_{d+1}) = w_1^2 + \ldots + w_{d+1}^2 + (w_1 + \ldots + w_{d+1})^2$, where $w(x_i) = w_i$ and x_i is a vertex of τ for $i = 1, \ldots, d+1$. Noting that $\|w\|^2_{L_2(\tau)} = \frac{\text{volume}(\tau)}{(d+1)(d+2)}(\sum_{i=1}^{d+1} w(x_i)^2 + [\sum_{i=1}^{d+1} w(x_i)]^2)$, equivalently we establish the following:

$$F(x_1, \ldots, x_d, 0) \leq \frac{d+1}{2} F(x_1, \ldots, , x_d, x_{d+1}), \quad \forall x_1, \ldots, x_{d+1}.$$

Assume there exist $x_1, \ldots, x_{d+1}$ such that $F(x_1, \ldots, x_d, 0) > \frac{d+1}{2} F(x_1, \ldots, , x_d, x_{d+1})$. Then the following equivalent expression leads to a contradiction.

$$0 > \frac{d-1}{2}(w_1^2 + \ldots + w_d^2) + \frac{d-3}{4}(w_1 + \ldots + w_d)^2 + (d+1)[w_{d+1} + (w_1 + \ldots + w_d)/2]^2.$$

For the sharp bound, observe that $F(x_1, \ldots, x_1, 0) = \frac{d+1}{2} F(x_1, \ldots, x_1, -x_1), \;\; x_1 \neq 0$. □

LEMMA 9.4. *The strengthened Cauchy-Schwarz inequality holds for all $\tau \in \tilde{\mathcal{T}}_i, v \in \tilde{\mathcal{S}}_i, w \in \tilde{\mathcal{S}}_j, j > i$:*

$$D(v, w)|_\tau \leq c \left(\frac{1}{\sqrt{2}}\right)^{j-i} |v|_{H^1(\tau)} \; 2^j \|w\|_{L_2(\tau)}, \tag{9.7}$$

where c is a constant depending only on the shape regularity.

Proof. We skip the details of the proof since they closely follow the 2D case in [7]. The major difference is the volume argument, where S denotes the strip in τ: $\frac{\text{volume}(S)}{\text{volume}(\tau)} = 1 - \left(1 - 3(\frac{1}{2})^{j-i}\right)^3 \leq 36 \left(\frac{1}{2}\right)^{j-i}$. Using the previous lemmas, the result holds with $c = 6\sqrt{2}\sqrt{c_0}$. □

Lemma 9.4 extends to A.7.2 in the following fashion. Summing over $\tau \in \tilde{\mathcal{T}}_i$ extends the local estimate (9.7) to the below global estimate:

$$D(v, w) \leq c \left(\frac{1}{\sqrt{2}}\right)^{(j-i)} |v|_{H^1(\Omega)} \; 2^j \|w\|_{L_2(\Omega)} \quad \forall v \in \tilde{\mathcal{S}}_i, \forall w \in \tilde{\mathcal{S}}_j, j > i \tag{9.8}$$

Consequently, the global estimate (9.8) holds for $v \in \mathcal{S}_i, w \in \mathcal{S}_j, j > i$, because $\mathcal{S}_i \subset \tilde{\mathcal{S}}_i, \;\; \mathcal{S}_j \subset \tilde{\mathcal{S}}_j$. A subtle requirement arises when (9.8) is generalized to

$$a(v, w) \leq c \left(\frac{1}{\sqrt{2}}\right)^{(j-i)} a(v, v) \; 2^j \|w\|_{L_2(\Omega)} \quad \forall v \in \tilde{\mathcal{S}}_i, \forall w \in \tilde{\mathcal{S}}_j, j > i.$$

The coefficient matrix p must be $C^1(\Omega)$ because of integration by parts. This is the main difficulty in extending the proof technology to $p \in L_\infty(\Omega)$.

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